

Push-Relabel Max Flow

Roadmap for rest of semester

"Average" analysis of algorithms

↑
(w/out sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

[illegible]

n increments $\Rightarrow O(n)$ total time
each increment takes $O(1)$ on average

" $O(1)$ amortized time"

- 1st* bit

- 2nd bit

- 3rd bit

4

- kth bit

how often total

every time n

every 2 times $\lfloor \frac{n}{2} \rfloor$

every 4 times $\lfloor \frac{n}{4} \rfloor$

every 2^{k-1} times $\lfloor \frac{n}{2^{k-1}} \rfloor$

Amortized Analysis

"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

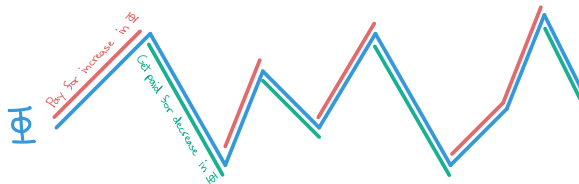
Multiple operations $OP_1, \dots, OP_K$

"OP₁ takes T_1 amortized time, OP₂ take $\approx$

OP_k takes T_k amortized time"

Potential method

Invent "potential Φ "



Amortized cost of operation will be

$$(\text{real time}) + \Delta\Phi$$

Credit schemes Idea: buy credits ahead of time to pay for work later

$$(\text{running time}) + \underbrace{O(1) \cdot (\Delta \text{ credit})}_{\text{net increase}} \leq \text{amortized time} \quad \text{put: } \begin{matrix} 1 & \text{credit on first bit} \\ \text{bit} & \end{matrix}$$

e.g. for binary counter, when incrementing:

when we need to flip bit i , we have already saved 1 credit to pay for it

put: 1 credit on first bit
 $\frac{1}{2}$ credit on 2nd bit
 $\frac{1}{4}$ credit on 3rd bit
 $\vdots$
 $+$ $\frac{1}{2^{i-1}}$ credits on i th bit

 2 credits total

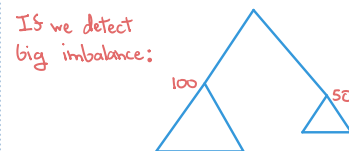
e.g. binary counter

$\Phi = \# \text{ bits set to } 1$

then for each inc,

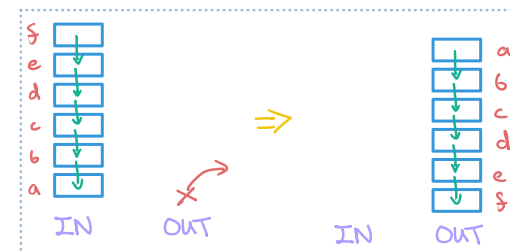
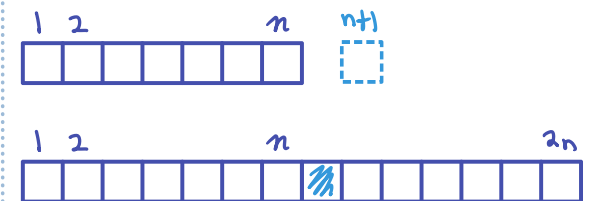
$$(\text{real time}) + \Delta \bar{\Phi} \leq O(1)$$

- leach bit reset to 0 paid by $\Delta \Phi$


$$\left(\begin{array}{c} \# \text{ in one} \\ \text{subtree} \end{array} \right) > 2 \times \left(\begin{array}{c} \# \text{ in other} \\ \text{subtree} \end{array} \right)$$

rebuild entire subtree tree optimally
(choose median to be root of subtree, recurse) (linear time)

Doubling Trick



today: push relabel algo. for max flow

- fastest algos we will see for flow

why?

(w/r/t class topics)

Roadmap for rest of semester

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① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

uses amortized analysis very nicely

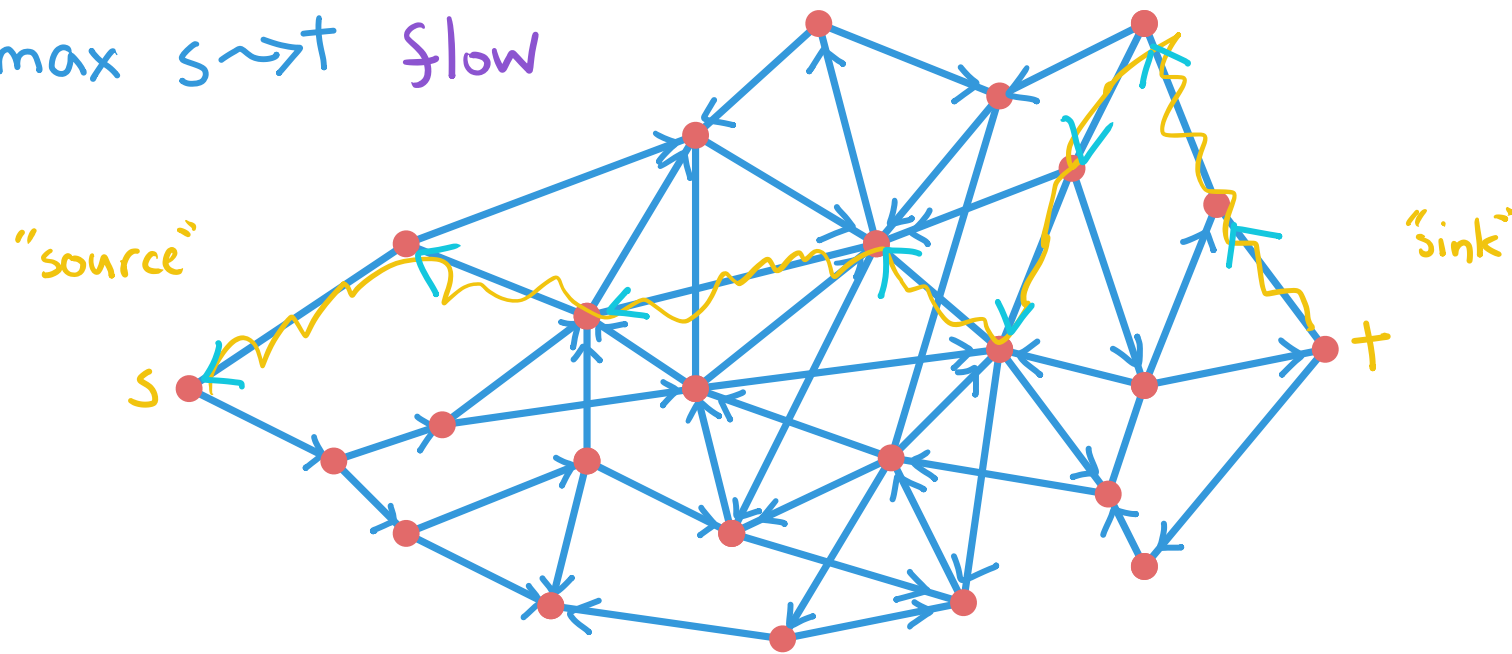
(gives us an example that is not a data structure)

(cleaned up)

(Edmonds, Karp)

How to compute

⑦ max $s \rightsquigarrow t$ flow



① while there is a path $p: s \rightsquigarrow t$

② take shortest path p and
route as much flow as possible

③ update residual graph

Analysis

Edmonds, Karp

① while there is a path $p: s \rightarrow t$

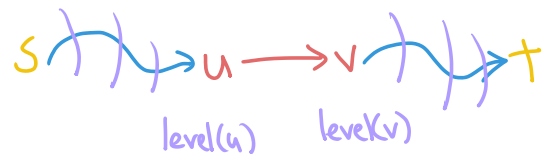
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Key

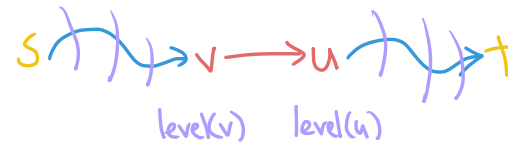
Lemma $\forall v \in V$, distance from s to v
in resid. graph never decreases

① e removed from ^(residual graph) RG,
 $\Rightarrow$ e in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(v) = \text{level}(u) + 1$$

② e = (u $\rightarrow$ v) added back in to RG
 $\Rightarrow$ $(v \rightarrow u)$ was in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(u) = \text{level}(v) + 1$$

each event ①/② $\Rightarrow$ $\text{level}(u)$, $\text{level}(v)$ have switched order.

level's nondecreasing by key lemma

$\Rightarrow$ e gets deleted/added at most n times

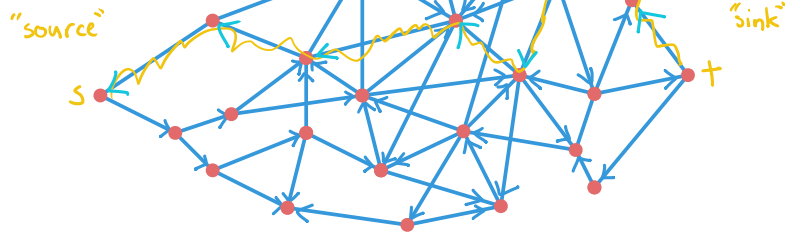
emphasis on distances in residual graph

(cleaned up)

(Edmonds, Karp)

How to compute

⑦ max $s \rightarrow t$ flow



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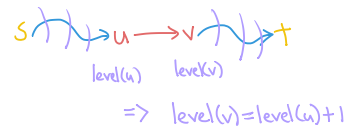
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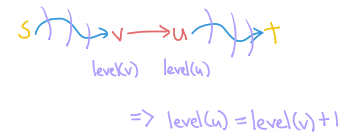
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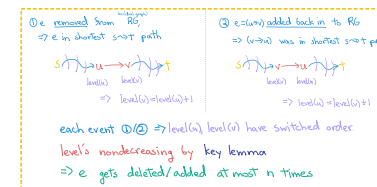
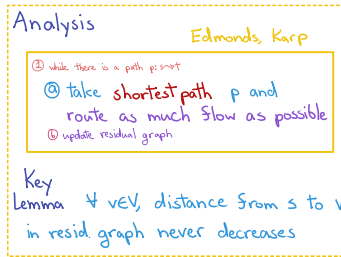
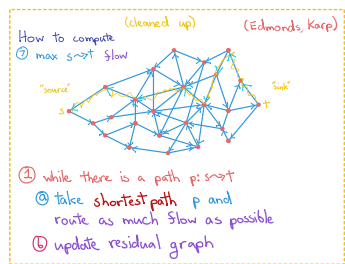
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emphasis on distances in residual graph



emphasis on distances in residual graph

Push-Relabel Framework

components:

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\nwarrow$ Relaxation of flows

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\nwarrow$ Relaxation of distances in residual graph

Push-Relabel Framework

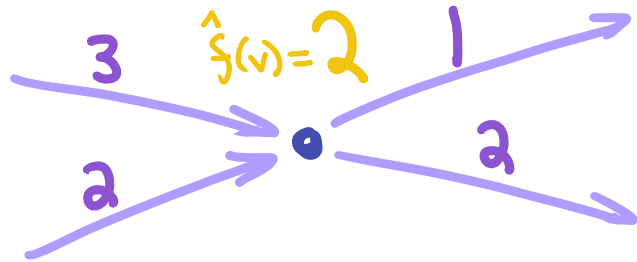
① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
↖ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u,v) - \sum_{e \in \mathcal{I}^+(v)} f(v,u)$



② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$
↖ Relaxation of distances in residual graph

Push-Relabel Framework

Residual graph $G_f = (V, E_f)$

just like for flows:

$$c_f(u, v) = c(u, v) - f(u, v) + f(v, u)$$

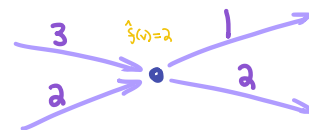
residual capacity

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

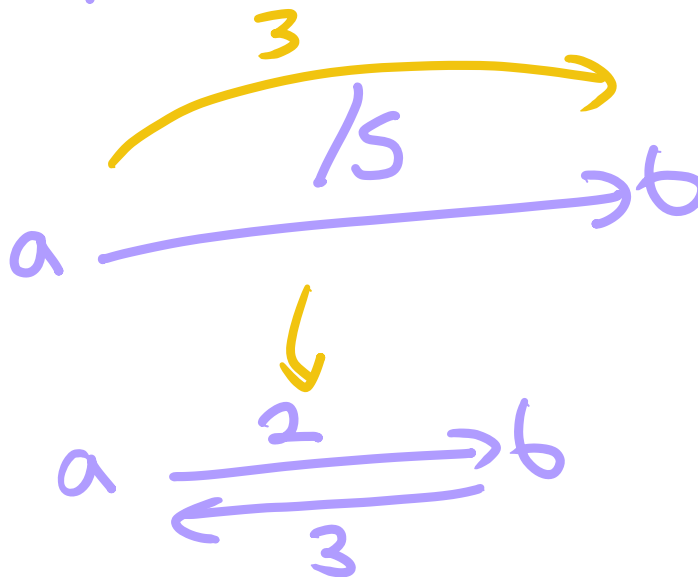
$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \vec{E}(v)} f(u, v) - \sum_{e \in \vec{E}^+(v)} f(v, u)$
net flow at v



② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

↗ Relaxation of distances in residual graph

① A "preflow" $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
 ~ Relaxation of flows
 • capacity constraints $0 \leq \hat{f}(e) \leq c(e) \quad \forall e \in E$
 • nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in J(v)} \hat{f}(u,v) - \sum_{e \in I(v)} \hat{f}(v,u)$
 net flow at v

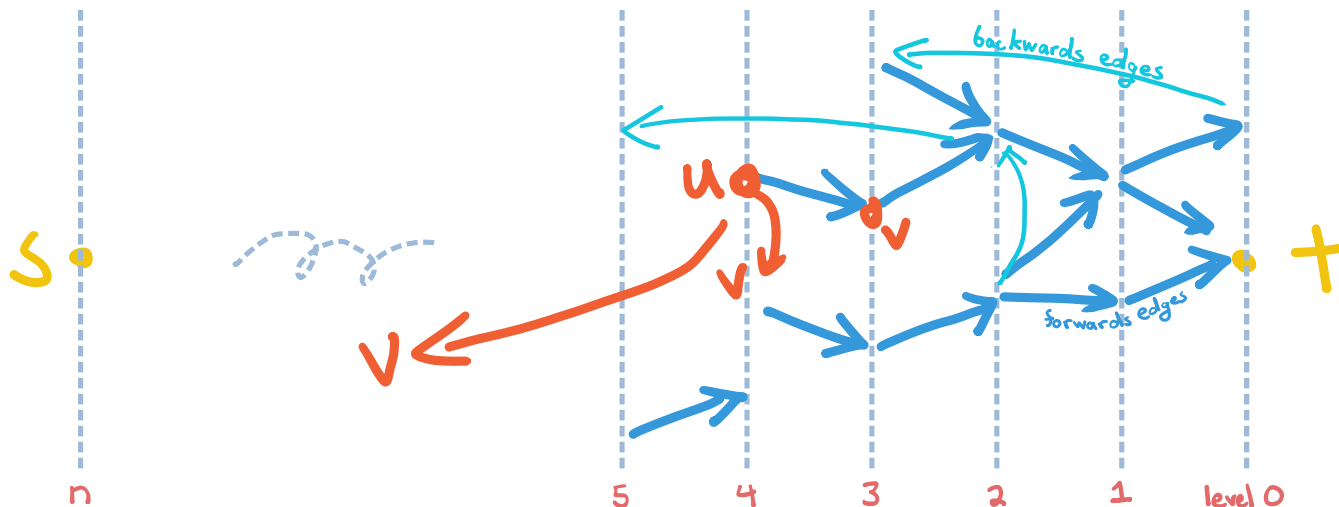
- Sink has level 0, source has level n

$$l(t) = 0 \quad l(s) = n$$

- edges (in resid. graph) go down ≤ 1 level

$$e = (u, v) \in E_f$$

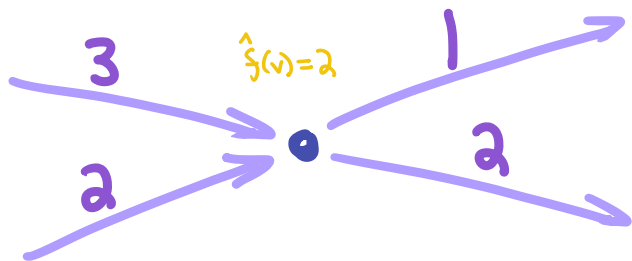
$$l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$$



Push-Relabel Framework

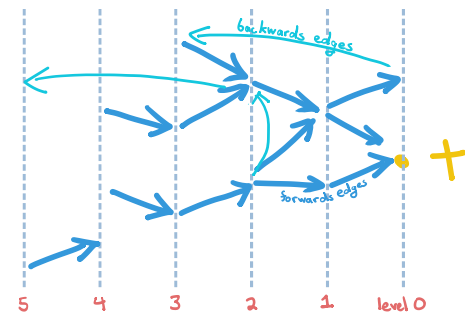
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Relaxation of flows

- capacity constraints
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net flow at v



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
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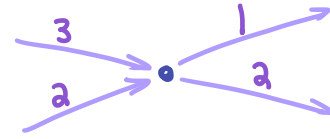
Initially: saturate all source edges, set all non-terminal levels to 0

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

$$\text{level}(s) = n \quad \text{level}(v) = 0 \quad \forall v \neq t$$

call a vertex u "active" if

$$\hat{f}(u) > 0$$



high-level goal:

push flow out of u "towards t "

Push-Relabel Operations

① Push flow along forward edge

② Relabel vertex to higher level

(and needs to follow the rules)

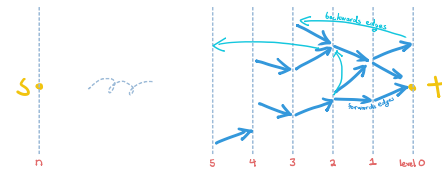
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Initially:

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

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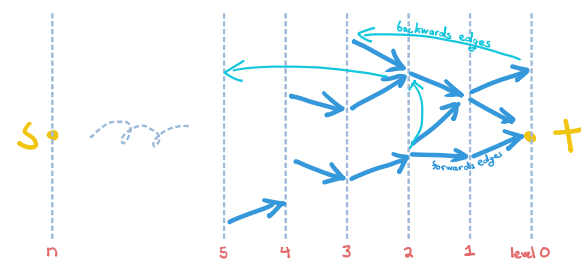
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

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① Push flow along forward edge

$$e = (u,v) \in E_f \text{ w/ } l(u) = l(v) + 1$$

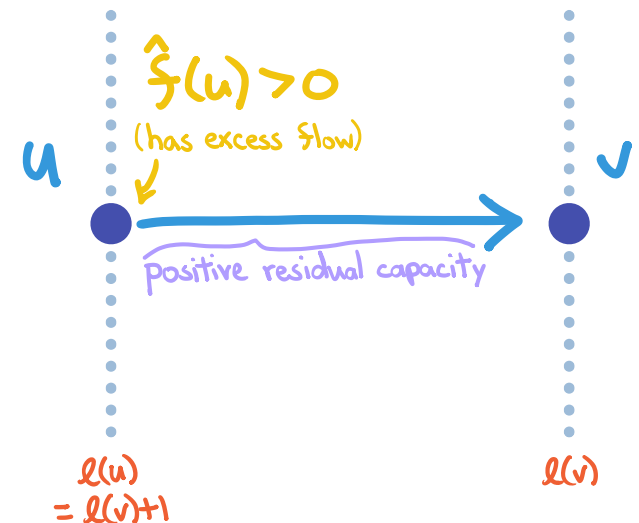
u active

(usually)

push as much flow as possible:

$$\min \{ c_f(u,v), \hat{f}(u) \}$$

(residual capacity) (excess flow)



② Relabel vertex to higher level

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
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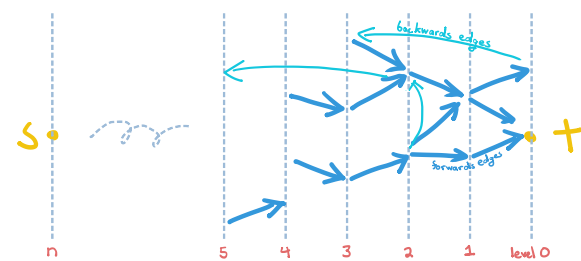
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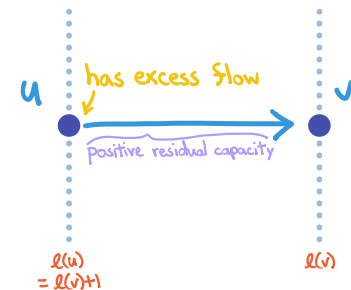
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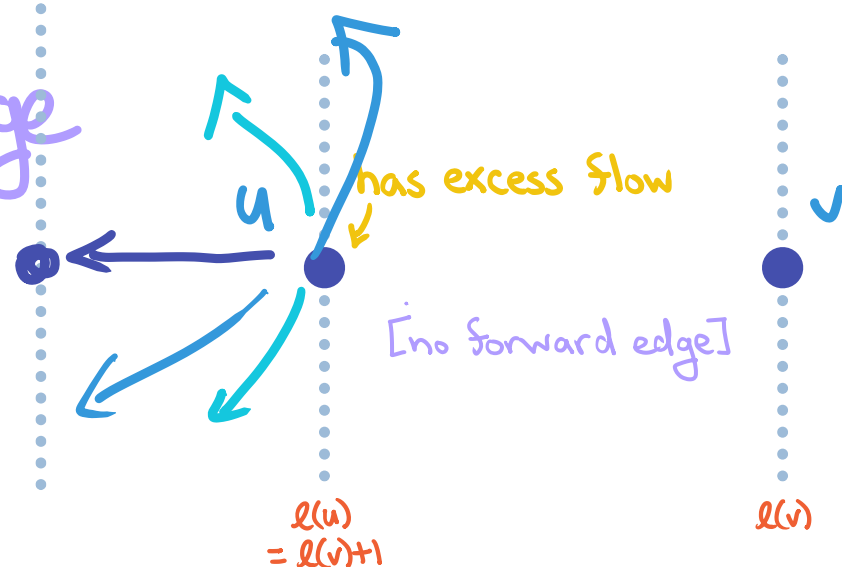
① Push flow along forward edge



② Relabel vertex to higher level

u active, no forward edge

safe to $l(u) = l(u) + 1$



Algorithm:

- ① while there is active $v \in V - \{s, t\}$
 - Ⓐ try to push along forward edge (u, v)
 - Ⓑ else relabel $l(v) = l(v) + 1$

WTF:

slow?

terminate?

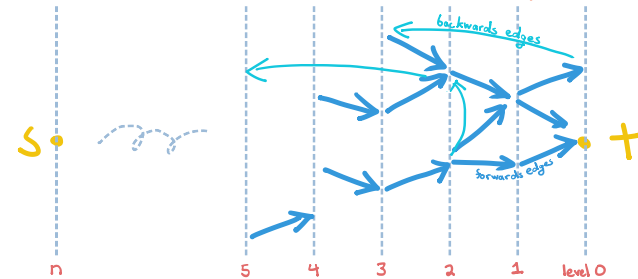
(s as t ?)

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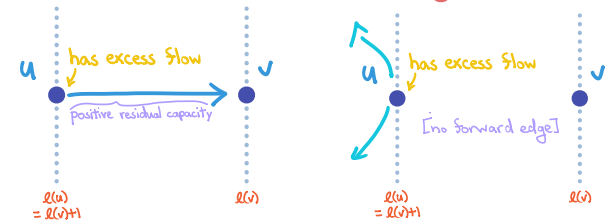
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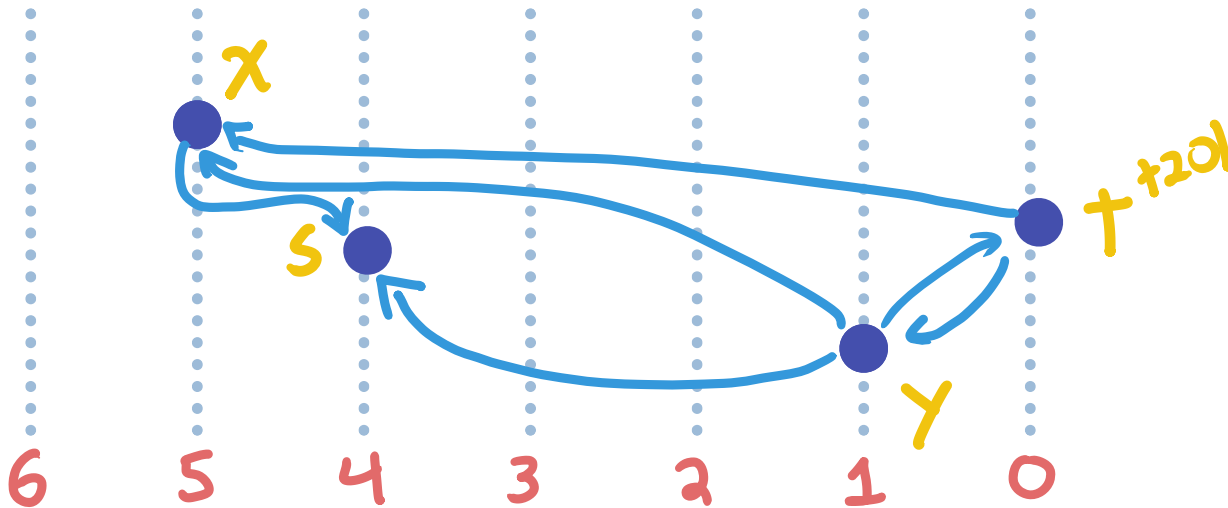
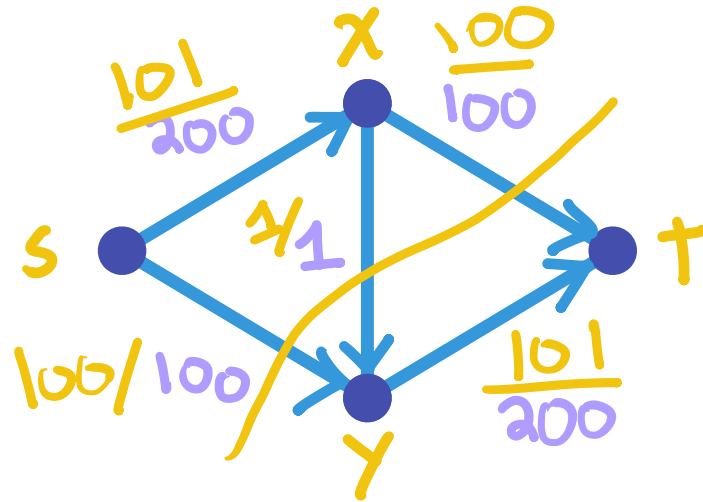
② Relabel vertex to higher level



① while there is active $v \in V - \{s, t\}$

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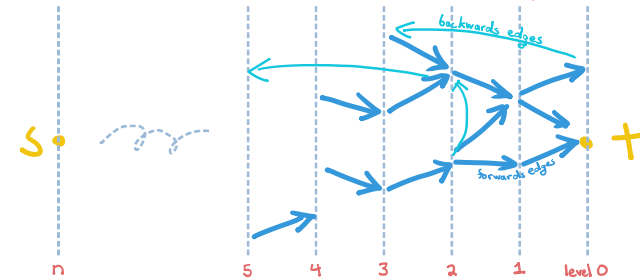
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Relaxation of distances in residual graph

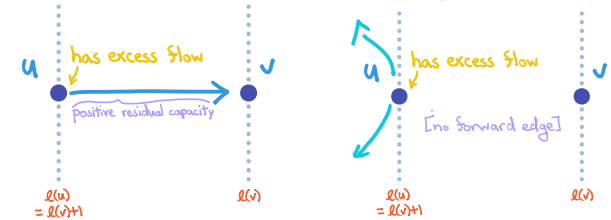
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WTF:

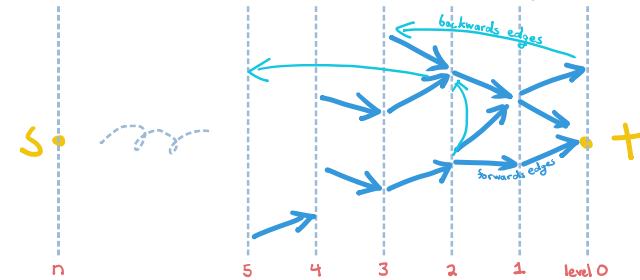
- flow? (let alone max flow?)
- running time?
- does it even terminate?

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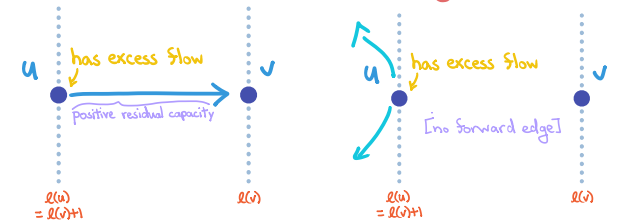
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① Push flow along forward edge

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Understanding "preflows"

Lemma let f be a preflow.
 then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

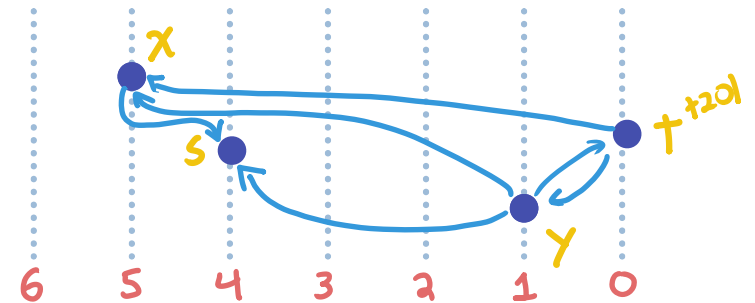
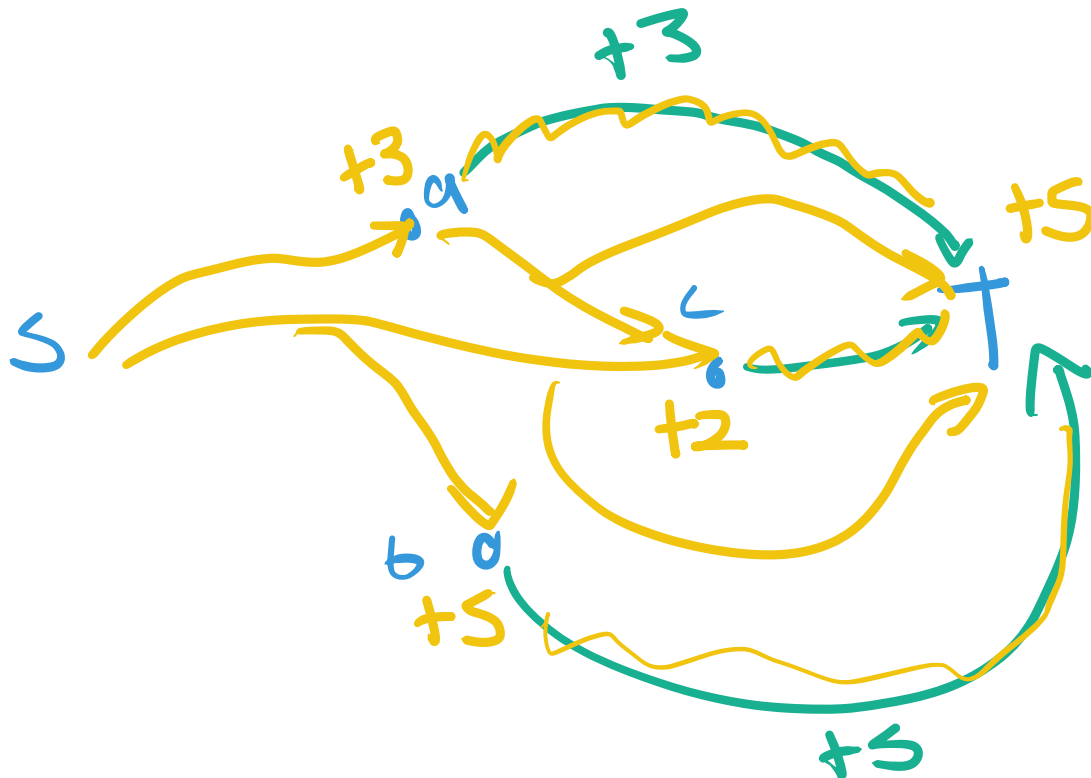
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Understanding "labels"

Lemma $\forall x, y \in V,$

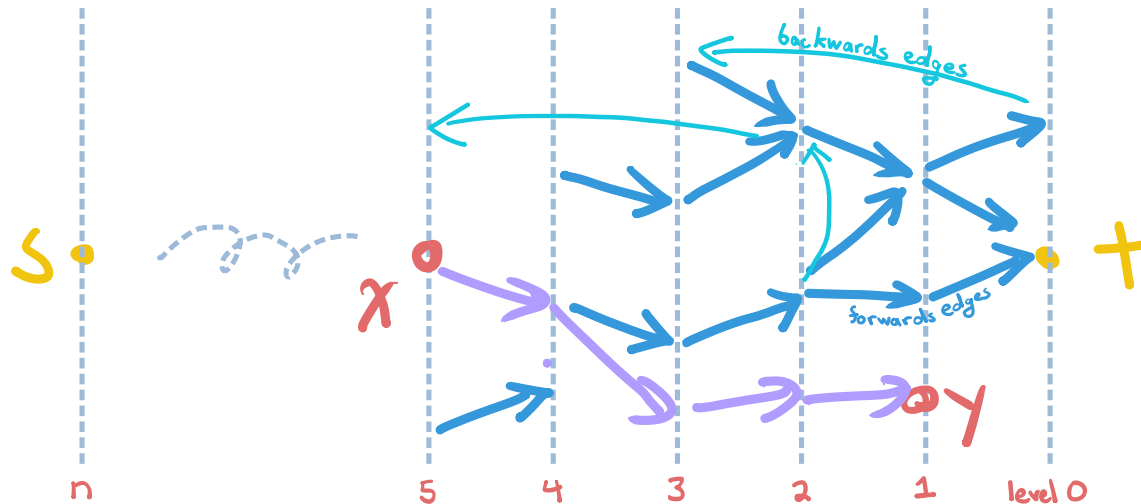
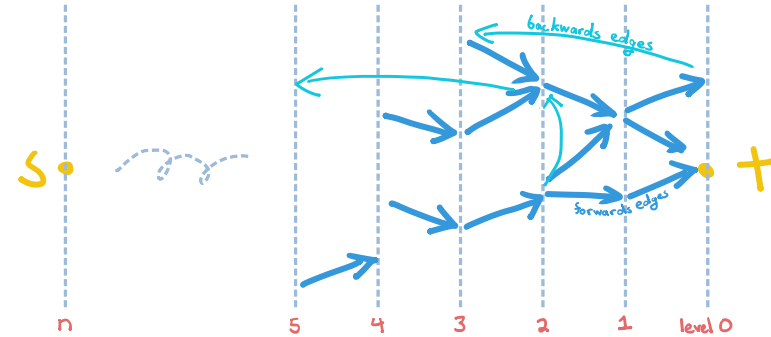
$$l(x) - l(y) \leq d(x, y)$$

where $d(x, y)$ = distance in resid. graph

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



Understanding "labels"

Lemma $\forall x, y \in V,$

$$l(x) - l(y) \leq d(x, y)$$

where $d(x, y)$ = distance in resid. graph

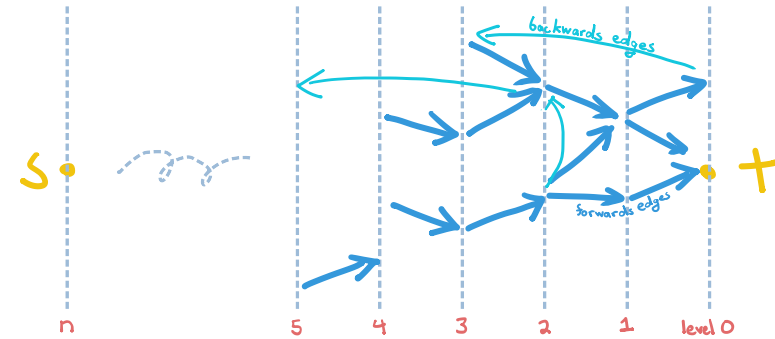
Consider $x=s, y=t$:

$$d(s, t) \geq n - 0 = n$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

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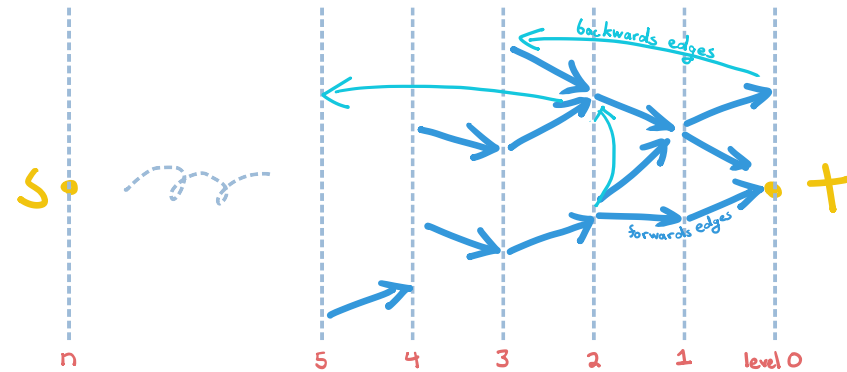
$$\Rightarrow d(s, t) \geq l(s) - l(t) = n$$

Corollary: No (s, t) -path in residual graph!

(always an (s, t) -cut in residual graph
separated by empty layers)

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

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Understanding "preflows"

Lemma let f be a preflow.
then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

Understanding "labels"

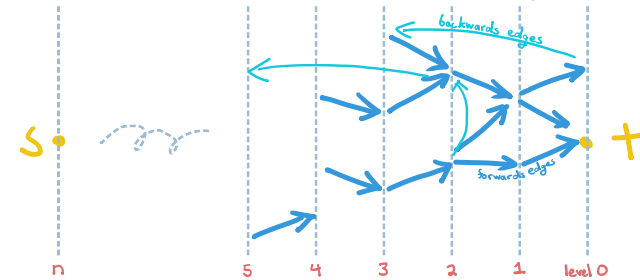
Lemma $\forall x, y \in V$,
 $l(x) - l(y) \leq d(x, y)$
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

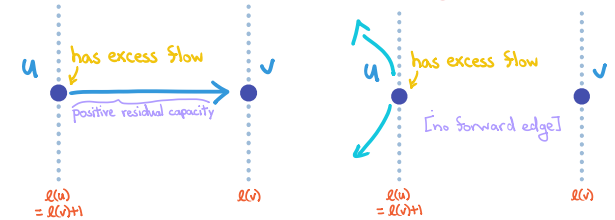
- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^+(v)} f(e, v) - \sum_{e \in \mathcal{I}^-(v)} f(v, e)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
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- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma if v active, then

$$l(v) < 2n$$

$$l(v) - l(s) \leq d(v, s)$$

$$\parallel \quad \parallel$$

$$n \quad \leq n-1$$

$$l(v) \leq 2n-1$$



Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

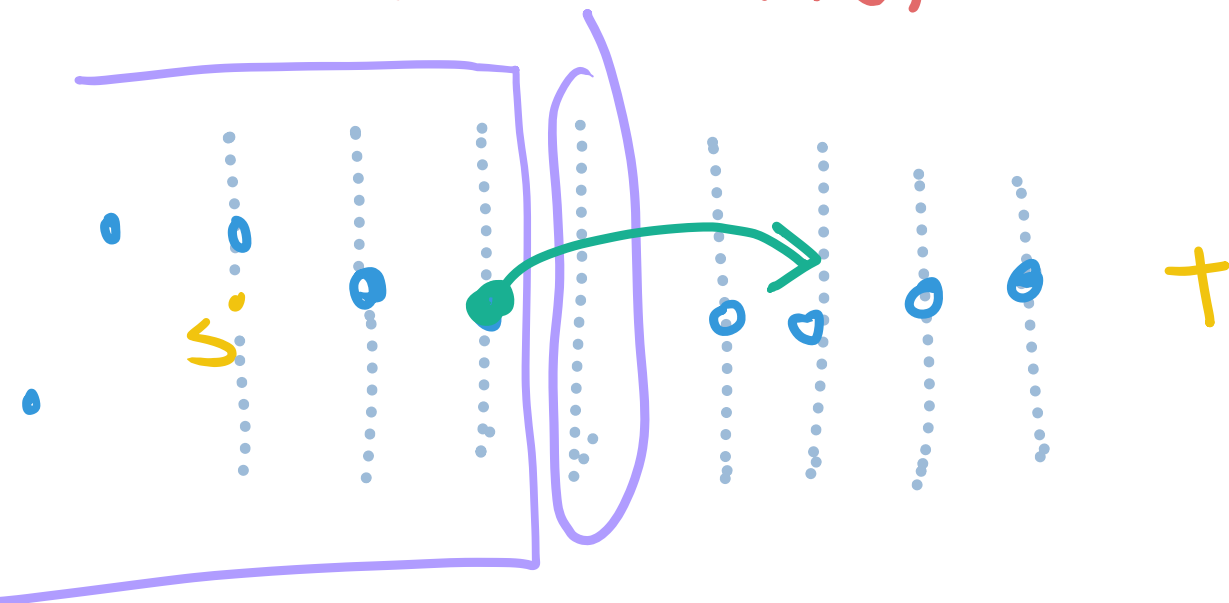
if f is a flow, then

① f is a max flow.

② there is empty layer $i \in \{1, \dots, n-1\}$

s.t. for $U = \{v: l(v) > i\}$,

$\partial^+(U)$ is a min (s, t) -cut

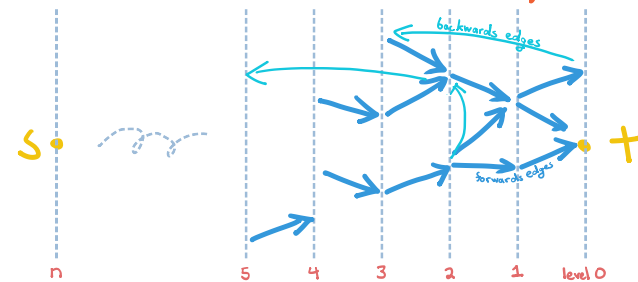


① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \partial^-(v)} f(e) - \sum_{e \in \partial^+(v)} f(e)$

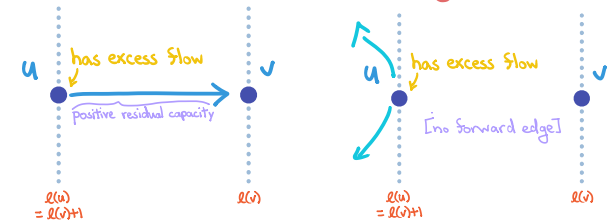
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

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(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

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 $\sim$ Relaxation of flows

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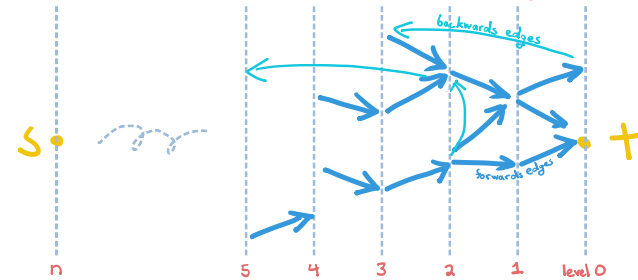
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

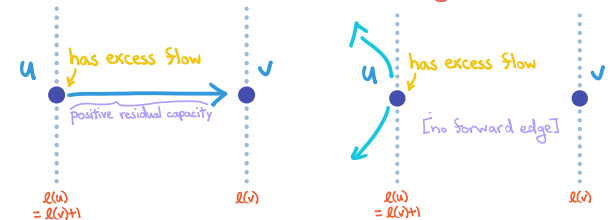
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① Push flow along forward edge

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Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

if f is a flow, then

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(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

when no v 's are active,

f is a max flow.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

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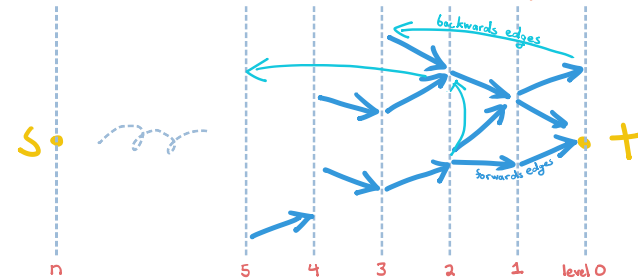
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

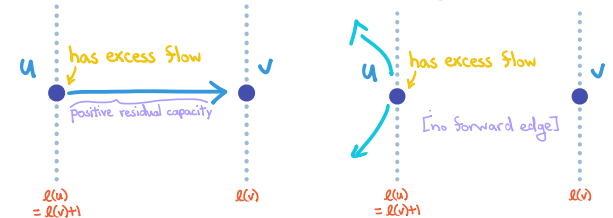
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① Push flow along forward edge

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(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Now we know why algo is correct.

next: running time

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq \hat{f}(e) \leq c(e) \quad \forall e \in E$$

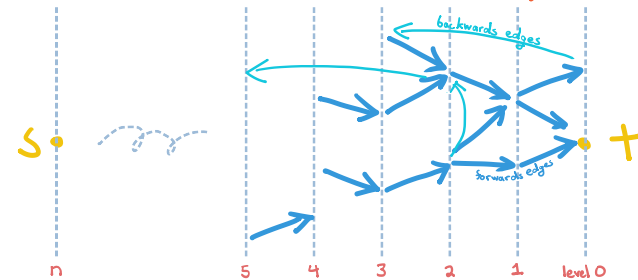
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② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

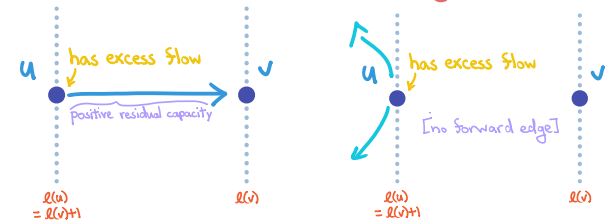
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① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Two kinds of pushes:

① Saturating push

uses all capacity of edge

② Non-saturating push

instead pushes out all excess flow from v

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

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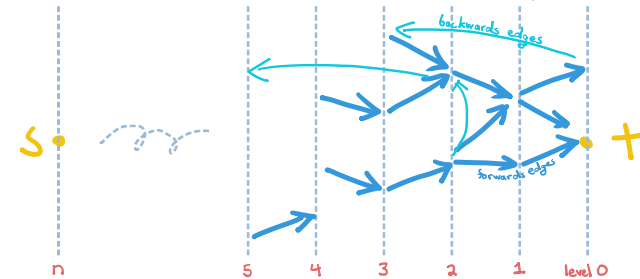
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② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$

$\sim$ Relaxation of distances in residual graph

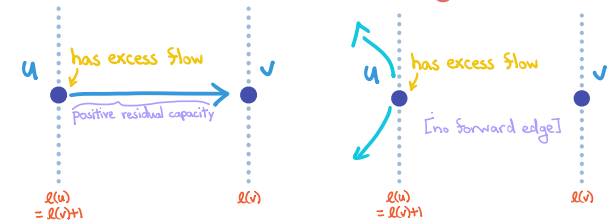
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① Push flow along forward edge

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(Algorithm)

- ① while there is active $v \in V - \{s, t\}$
 - Ⓐ try to push flow out of v
 - Ⓑ otherwise relabel v

we will bound:

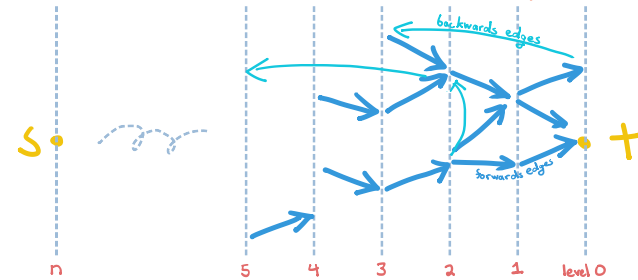
- ① # relabelings
- ② # saturating push
- ③ # non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

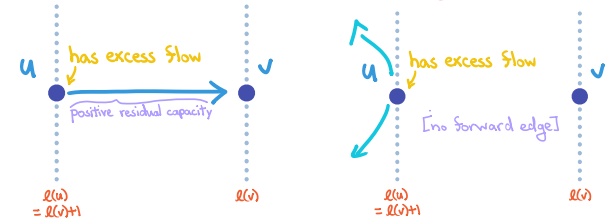
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- ① Push flow along forward edge
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Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

$$l(v) \leq 2n$$

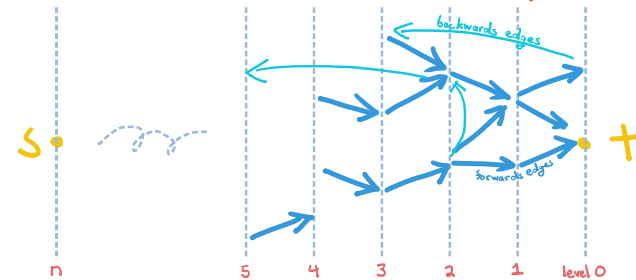
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Relaxation of flows

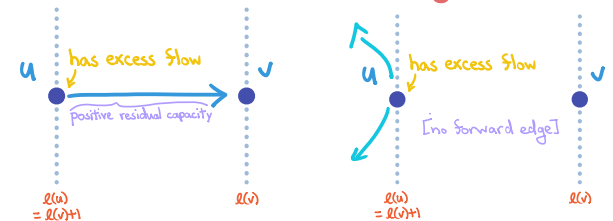
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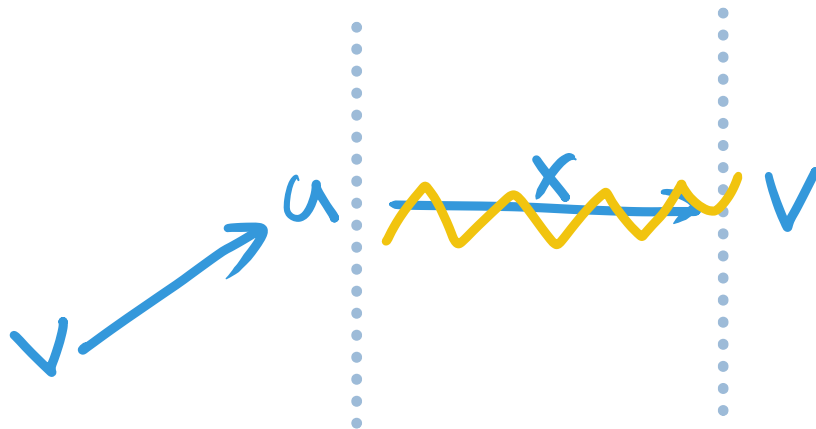
- ① Push flow along forward edge
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Lemma

(# saturating pushes) $\leq O(mn)$

($O(n)$ per edge)



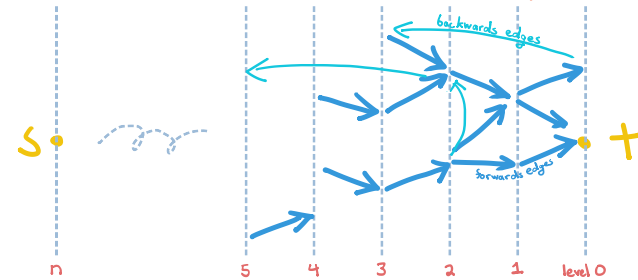
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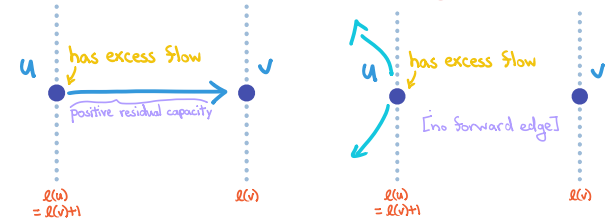
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- ① Push flow along forward edge
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Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

$$\Phi = \sum_{v \text{ active}} l(v)$$

relabel:

$$\Delta \Phi = +1$$

saturating push:

$$\Delta \Phi \leq 2n$$

non-saturating push

$$\Delta \Phi \leq -1$$

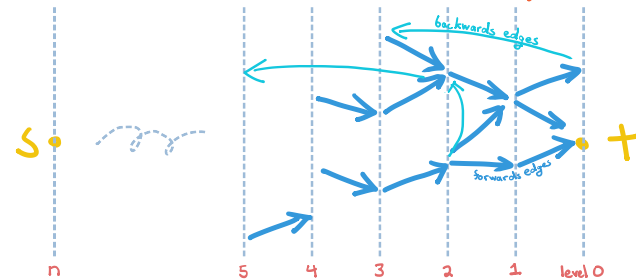
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Relaxation of flows

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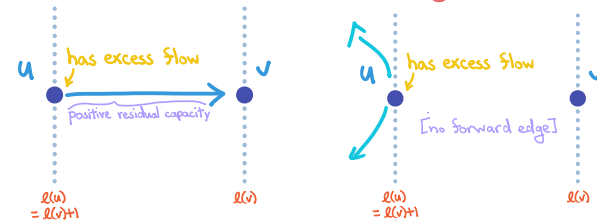
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① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

$$(\# \text{ relabels}) \leq O(n^2)$$

$$(\# \text{ saturating pushes}) \leq O(mn)$$

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

$$O(mn^2)$$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
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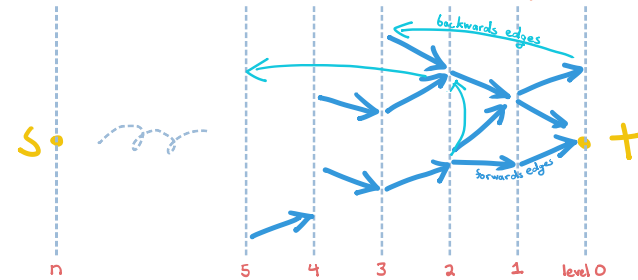
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

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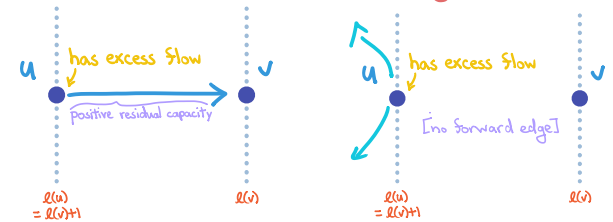
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

$O(mn^2)$ time

faster?

Bottleneck was non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

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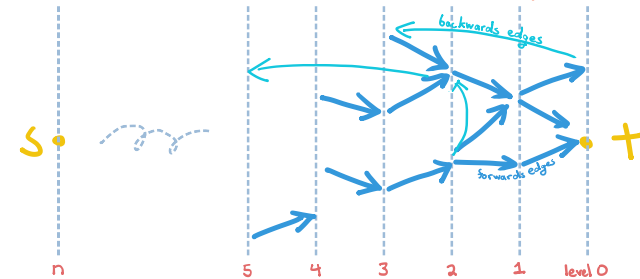
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Relaxation of distances in residual graph

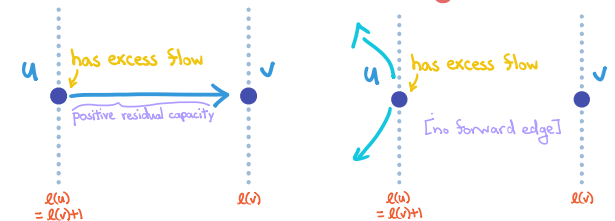
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① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

① while there is active $v \in V - \{s, t\}$

★ let $v =$ active w/ highest level

@ try to push flow out of v

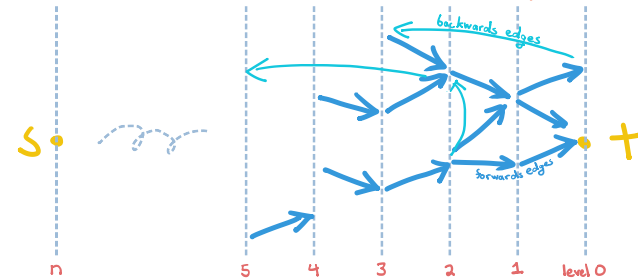
⑥ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
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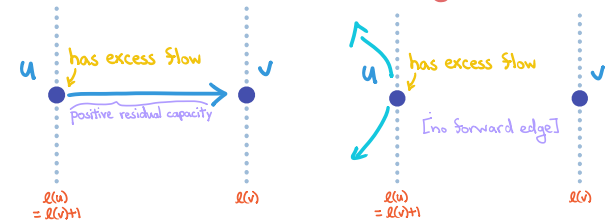
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① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v =$ active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2) \quad \checkmark$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn) \quad \checkmark$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

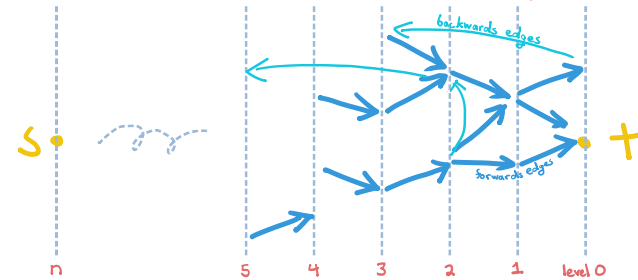
↑ better?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

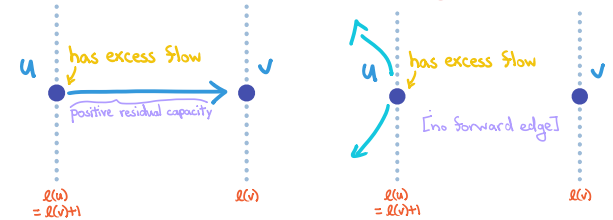
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$

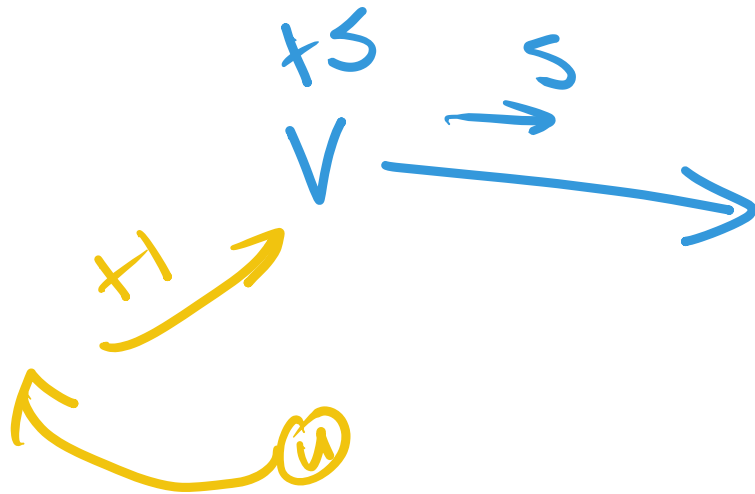


- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$



Top-down push-relabel

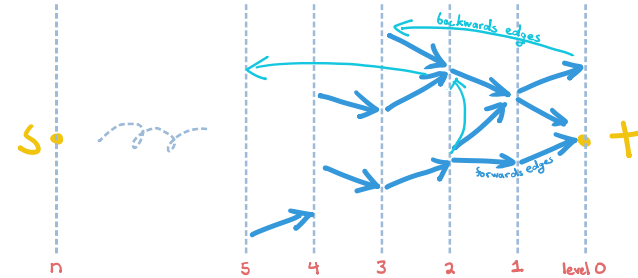
- ① while there is active $v \in V - \{s, t\}$
- ② let $v =$ active w/ highest level
- ③ try to push flow out of v
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

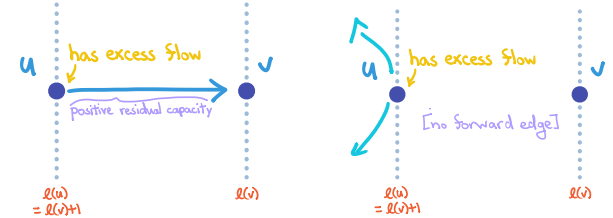
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Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v = \text{active w/ highest level}$
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

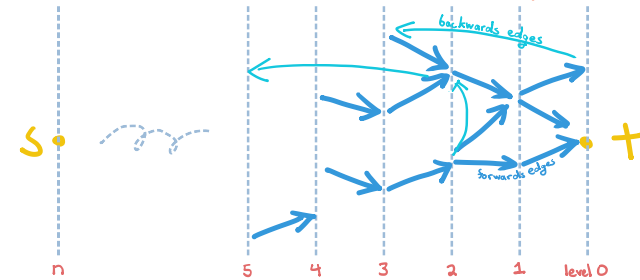
$\Rightarrow O(n^3)$ time overall.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

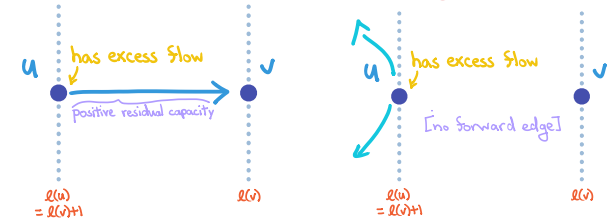
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Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^2 \sqrt{m})$$

$$\Phi = \sum_{v: \text{active}} |\{w: l(w) \leq l(v)\}|$$

$$\text{relabel: } \Delta \Phi \leq n$$

$$\text{sat. push: } \Delta \Phi \leq n$$

$$\text{total } \uparrow \Phi \leq mn^2$$

Top-down push-relabel

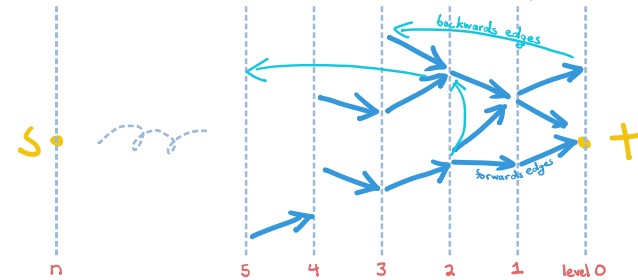
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
~ Relaxation of flows

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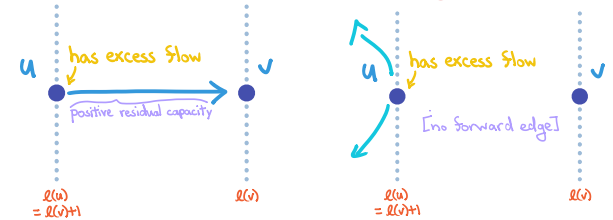
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① Push flow along forward edge

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non-saturating push at v

$$\Delta \Phi = -(\# w: \ell(w) = \ell(v))$$

"small" $< \sqrt{m}$

"big" $> \sqrt{m}$

Top-down push-relabel

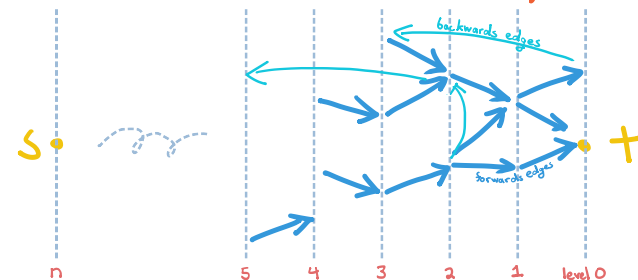
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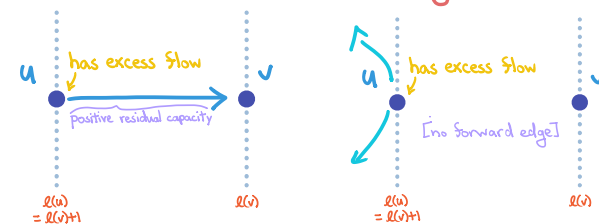
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Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^2 \sqrt{m})$$

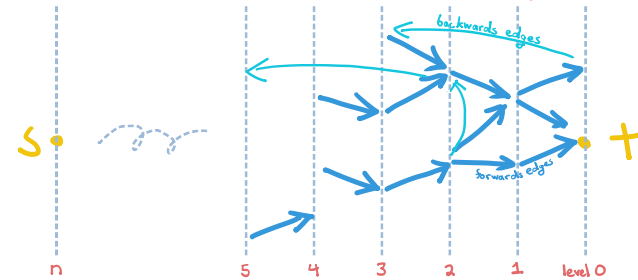
$$\Rightarrow O(n^2 \sqrt{m}) \text{ time}$$

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 $\sim$ Relaxation of flows

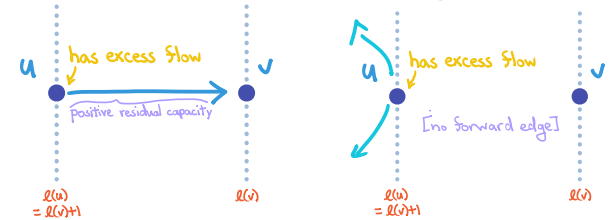
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I should also mention

"scaling" version

$O(mn \log U)$ time

for integer capacities between
1 and U .

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

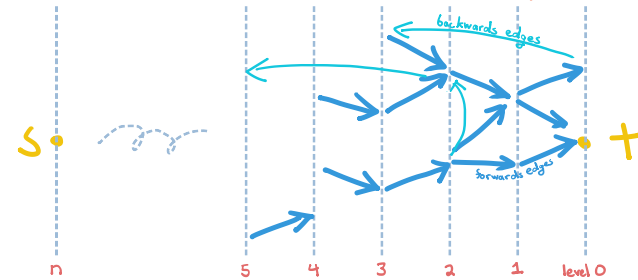
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