Lazy Data Structures

Roadmap for rest of semester

"Average" analysis of algorithms

(w/out sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

Incrementing Binary Counters

```
i++;        //increment i
```

In bits:

0, 1, 10, 11, 100, 101, ...

11111111111111111111111111111111

100000000000000000000000000000000

k bits ⇒ k flips

From i = 1 to n, how many total flips

*least significant

	how often	total
• 1st* bit	every time	n
• 2nd bit	every 2 times	$\lfloor \frac{n}{2} \rfloor$
• 3rd bit	every 4 times	$\lfloor \frac{n}{4} \rfloor$
⋮		
• kth bit	every 2^{k-1} times	$\lfloor \frac{n}{2^{k-1}} \rfloor$

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n increments $\Rightarrow$ $O(n)$ total time

each increment takes $O(1)$ <u>on average</u>

"$O(1)$ amortized time"

Amortized Analysis

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"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

Multiple operations $OP_1, \sim, OP_k$

"OP_1 takes T_1 amortized time, OP_2 take $\sim$

OP_k takes T_k amortized time"

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Ways to do amortized analysis

- direct analysis (like above)
- credit schemes ("banker method")
- potential function method

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"OP_1 takes T_1 amortized time, OP_2 take … OP_k takes T_k amortized time"

Credit schemes

Idea: buy credits ahead of time to pay for work later

$$(\text{running time}) + O(1) \cdot (\underbrace{\Delta \text{ credit}}_{\text{(net increase)}}) \leq \text{amortized time}$$

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put: 1 credit on first bit
$\frac{1}{2}$ credit on 2nd bit
$\frac{1}{4}$ credit on 3rd bit
⋮
+ $\frac{1}{2^{i-1}}$ credits on ith bit

---

2 credits total

when we need to flip bit i, we have already saved 1 credit to pay for it

Credit schemes

Idea: buy credits ahead of time to pay for work later

$$(\text{running time}) + O(1)\cdot(\underbrace{\Delta\ \text{credit}}_{\text{(net increase)}}) \le \text{amortized time}$$

Main rule:

Only spend credits that you already saved

e.g. for binary counter, when incrementing:

0, 1, 10, 11, 100, 101, ~

111111…111

1000000…000

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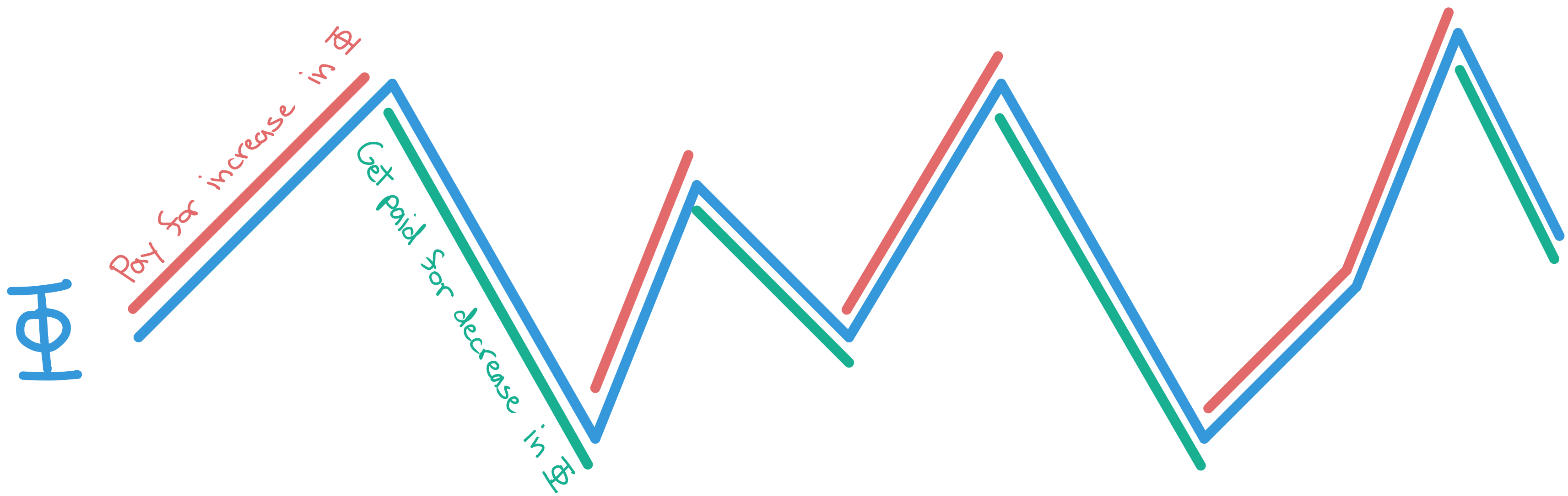
Potential method

Invent "potential Φ"



Amortized cost of operation will be

(real time) + $\Delta\Phi$

(change in potential)

Potential method

Amortized cost of operation will be

$$(\text{real time}) + \Delta\Phi$$

(change in potential)

e.g. binary counter

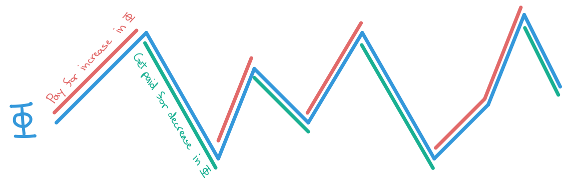
Φ = # bits set to 1

then for each inc,

$$(\text{real time}) + \Delta\Phi \leq O(1)$$

(each bit reset to 0 paid by $\Delta\Phi$)

"potential Φ"



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Potential method

Invent "potential Φ"

Pay for increase in Φ

Get paid for decrease in Φ

"Amortized cost of operation will be"

(real time) + $\Delta\Phi$
(change in potential)

why? suppose n operations

T_i = time of ith operation

Φ_i = Φ after i operations

$$\Delta\Phi_i = \Phi_i - \Phi_{i-1}$$

$$\sum_{i=1}^{n} (T_i + \Delta\Phi_i) = (\text{total time}) + \overset{\geq 0}{\Phi_n}$$

$$\geq (\text{total time})$$

$$(\text{total time}) \leq \sum_{i=1}^{n} (T_i + \Delta\Phi_i)$$

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⋮

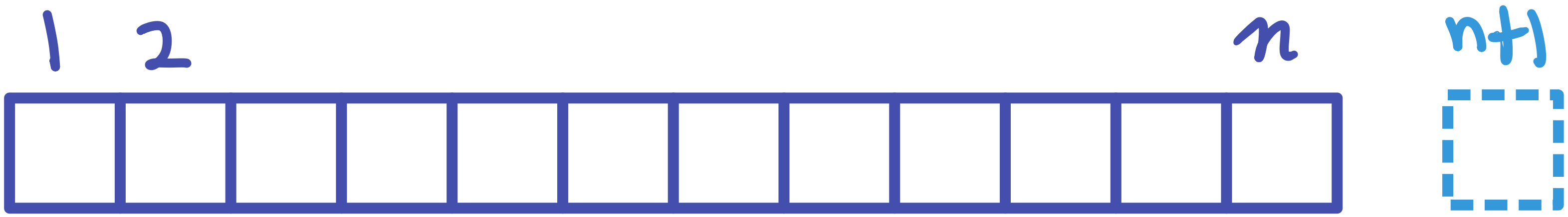
+ $\frac{1}{2^{i-1}}$ credits on ith bit

2 credits total

- potential function method

$\underline{\Phi}$ = # bits set to 1

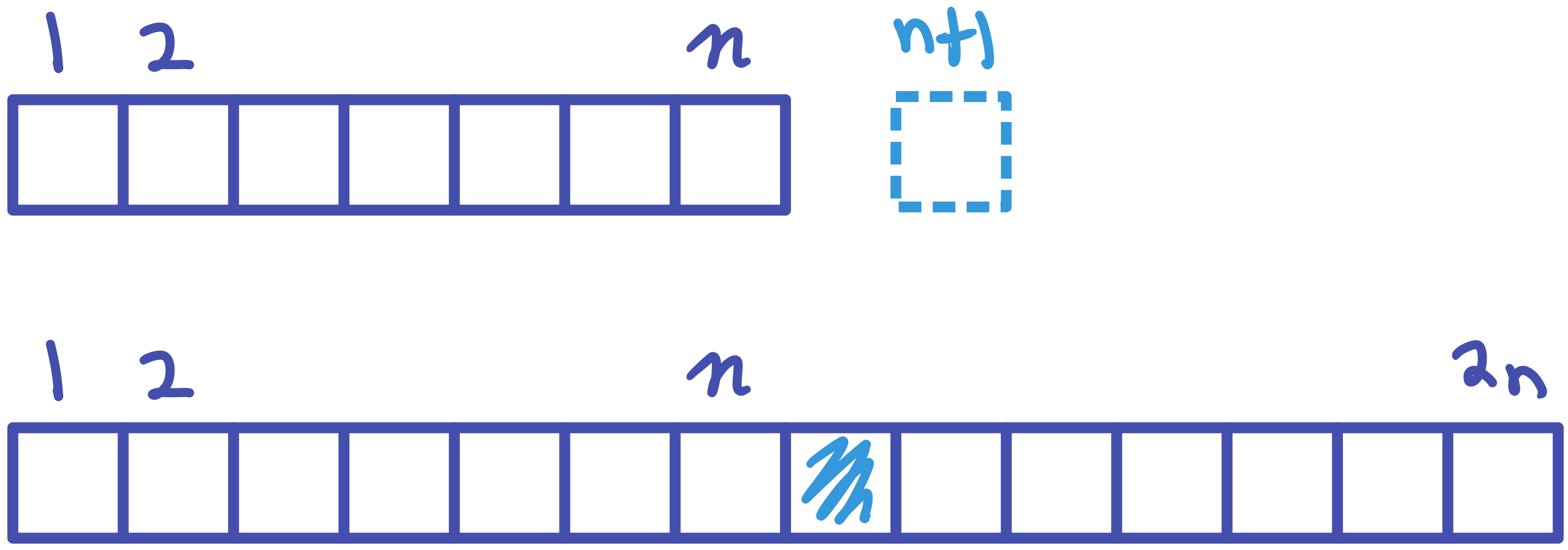
Array-backed lists



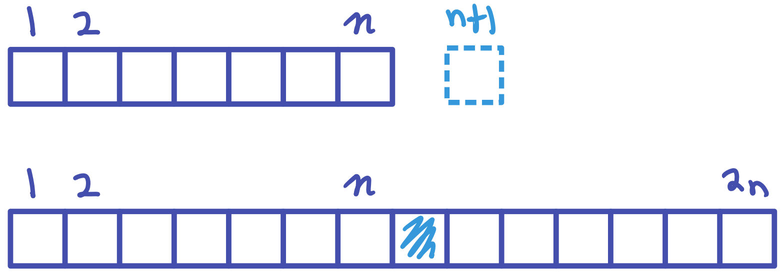
Good: O(1) acces by index

Bad: O(n) append

Doubling Trick



Doubling Trick



Theorem

w/ doubling trick, arraylist has

- lookup in $O(1)$
- append in $O(1)$ amortized

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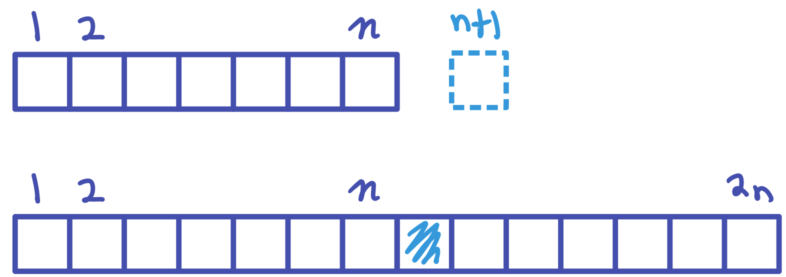
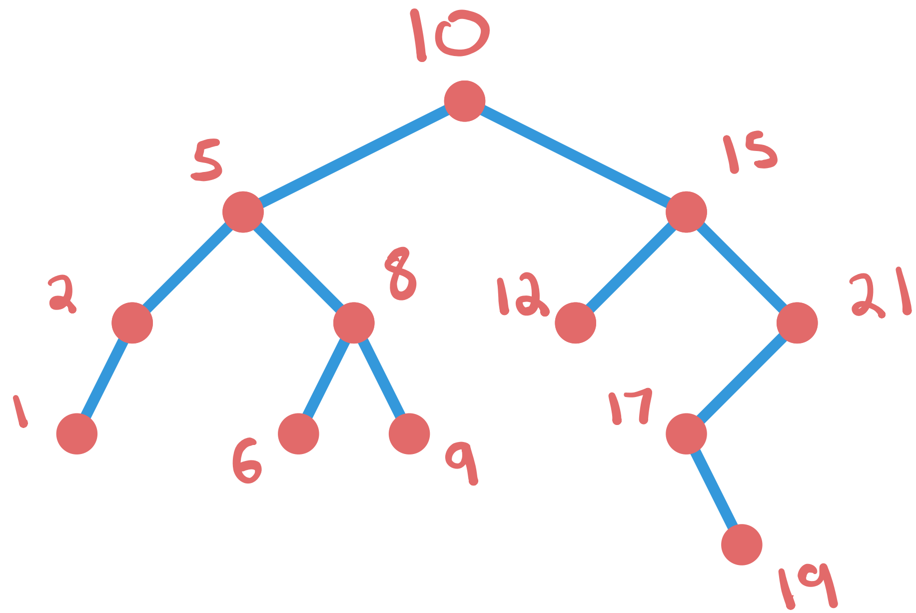
Proof

Doubling Trick



Search trees

organizes keys
in a tree



Invariant:

a node's # is

- $\geq$ any # in left subtree
- $\leq$ any # in right subtree

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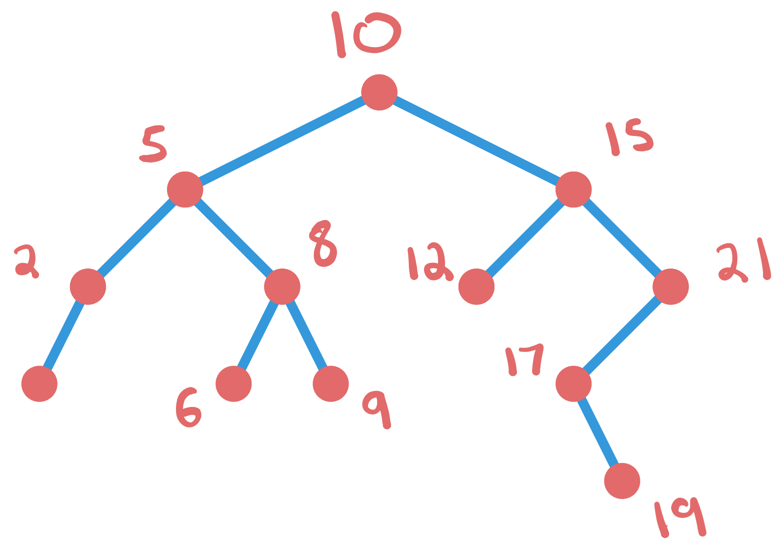
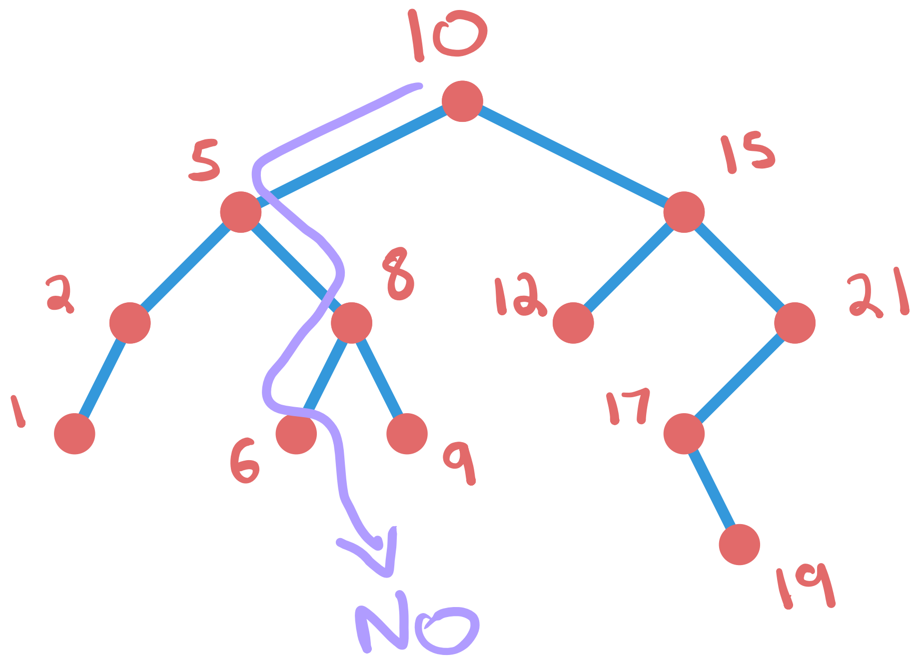


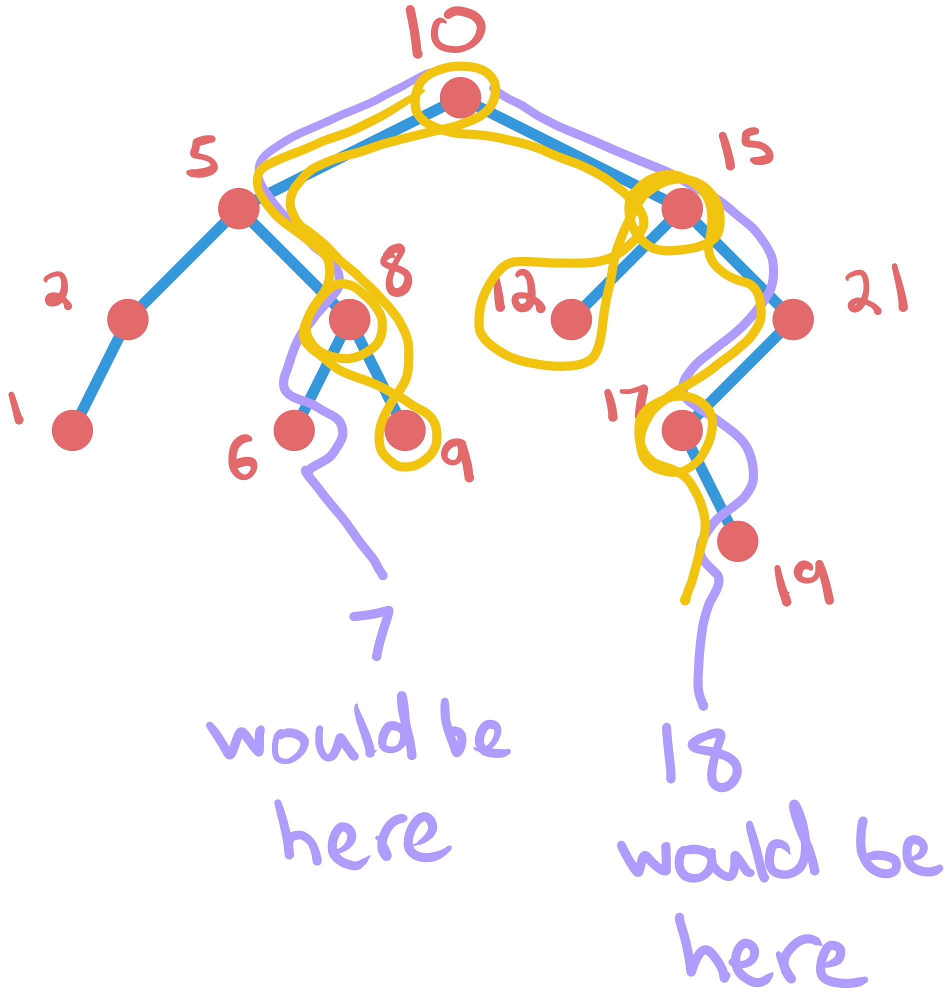
Leads to operations like:

- is a given key in the set?
- what is the first key $\geq x$
- list all keys between x and y
- how many keys are between x and y

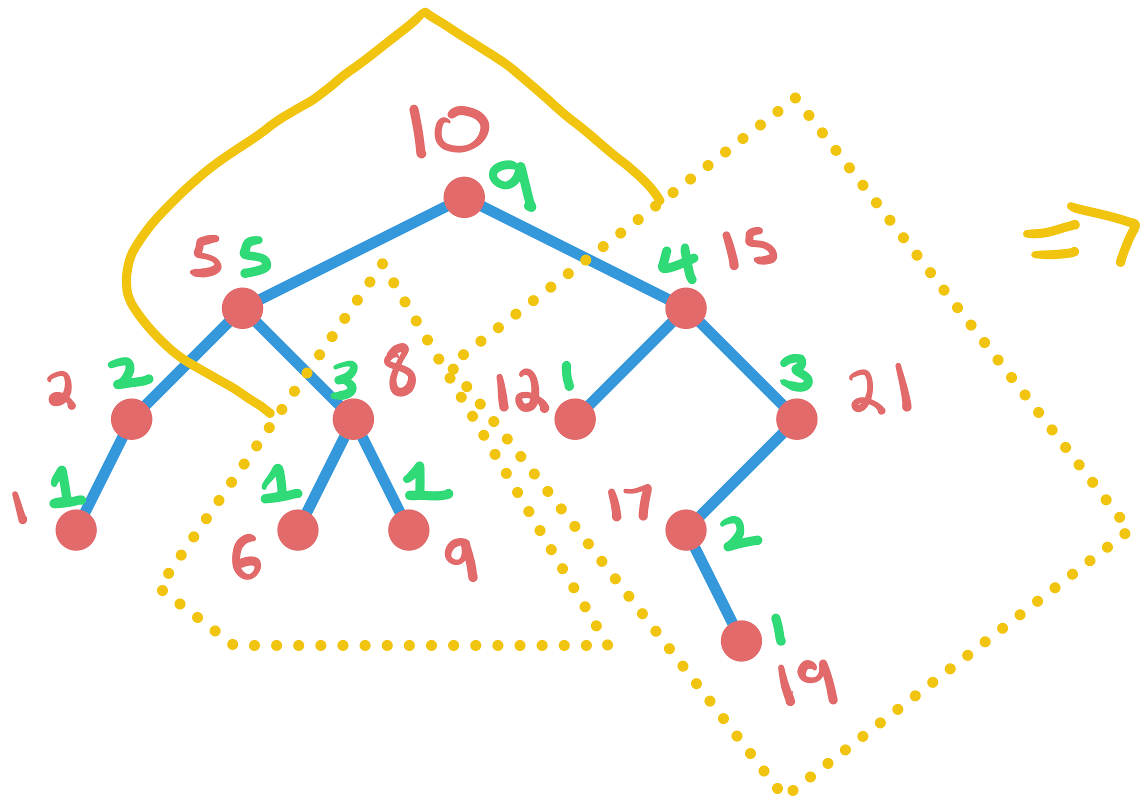
does the tree contain 7?



list all keys between 7 and 18

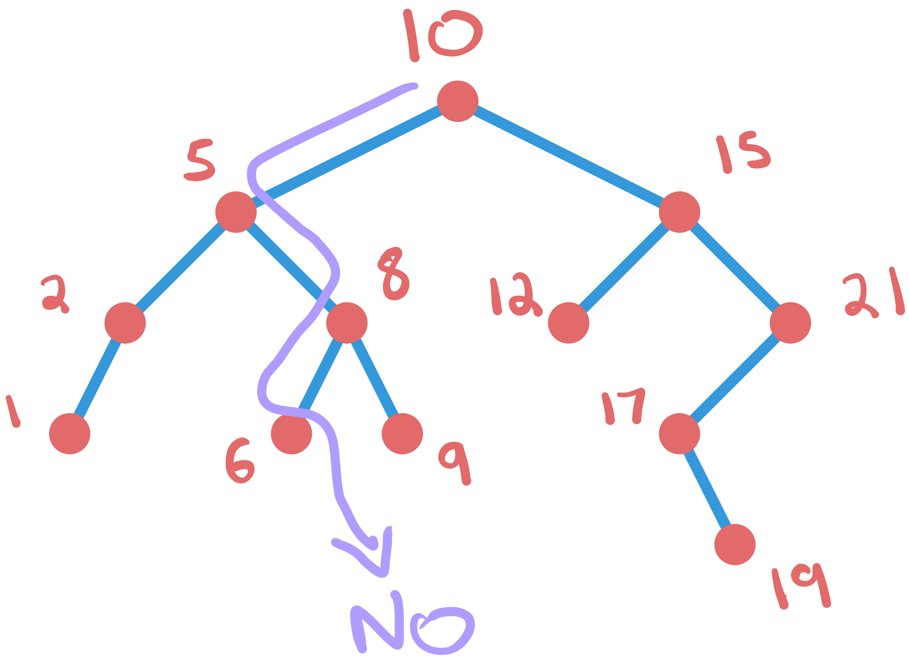


how many keys are between 5 and 25

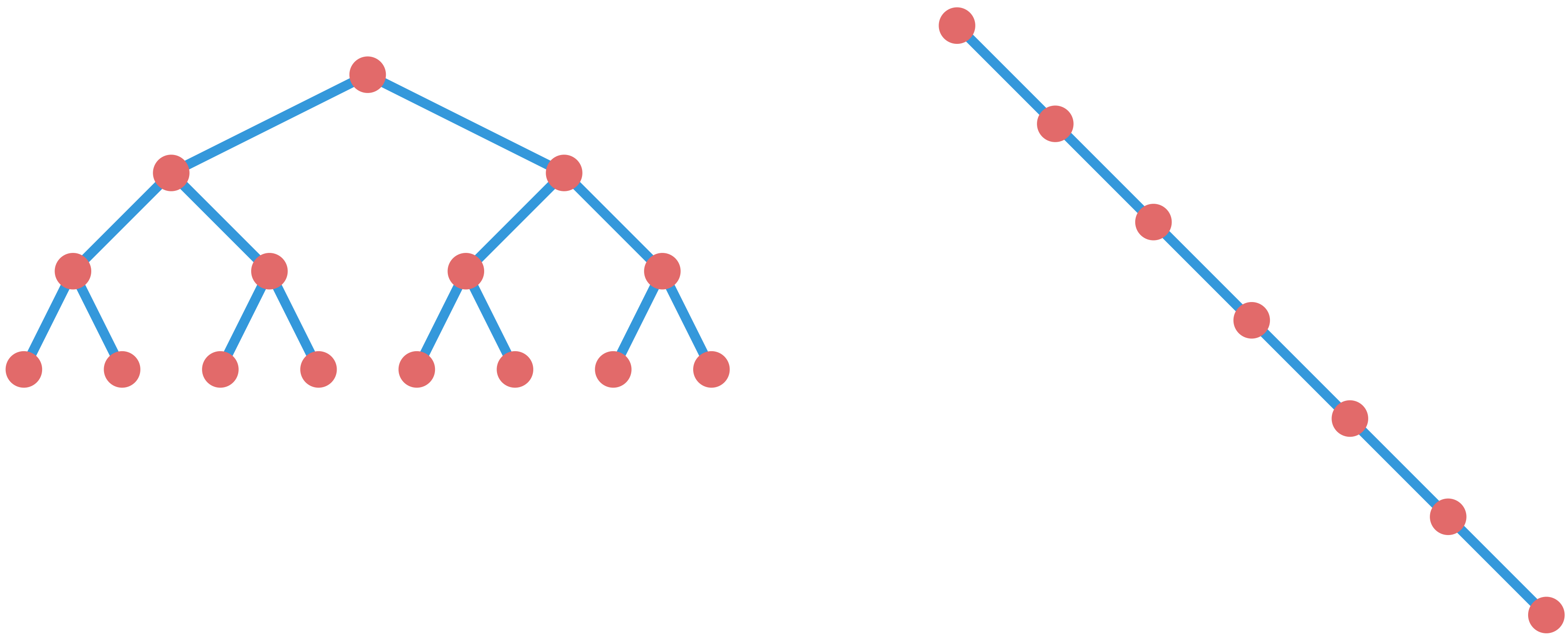


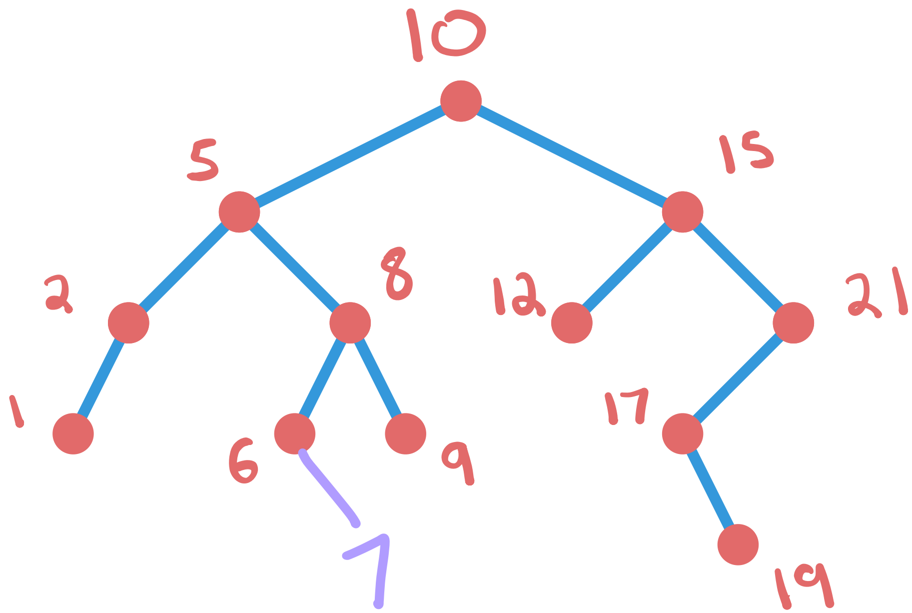
$\Rightarrow 9$

if we write down # nodes in each subtree, then faster



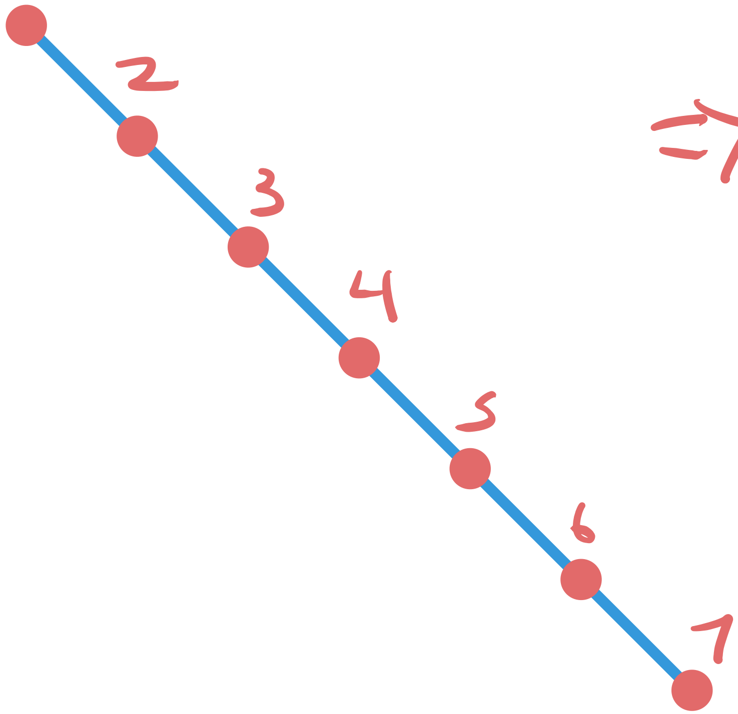
operations take time proportional to the height of a tree



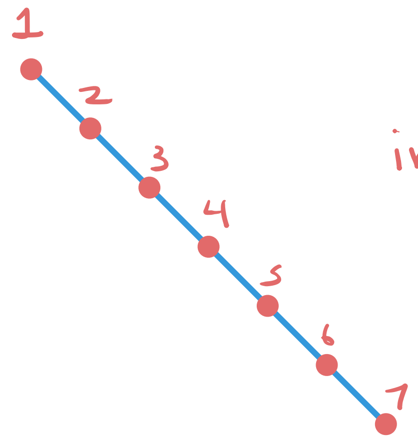


Most natural method for insertion is as leaves (search for where the key would be, and put it there)

1 insert 1, 2, 3, 4, 5, 6, 7



$\Rightarrow$ imbalanced



insert 1, 2, 3, 4, 5, 6, 7

$\Rightarrow$ imbalanced

How to keep a tree balanced?

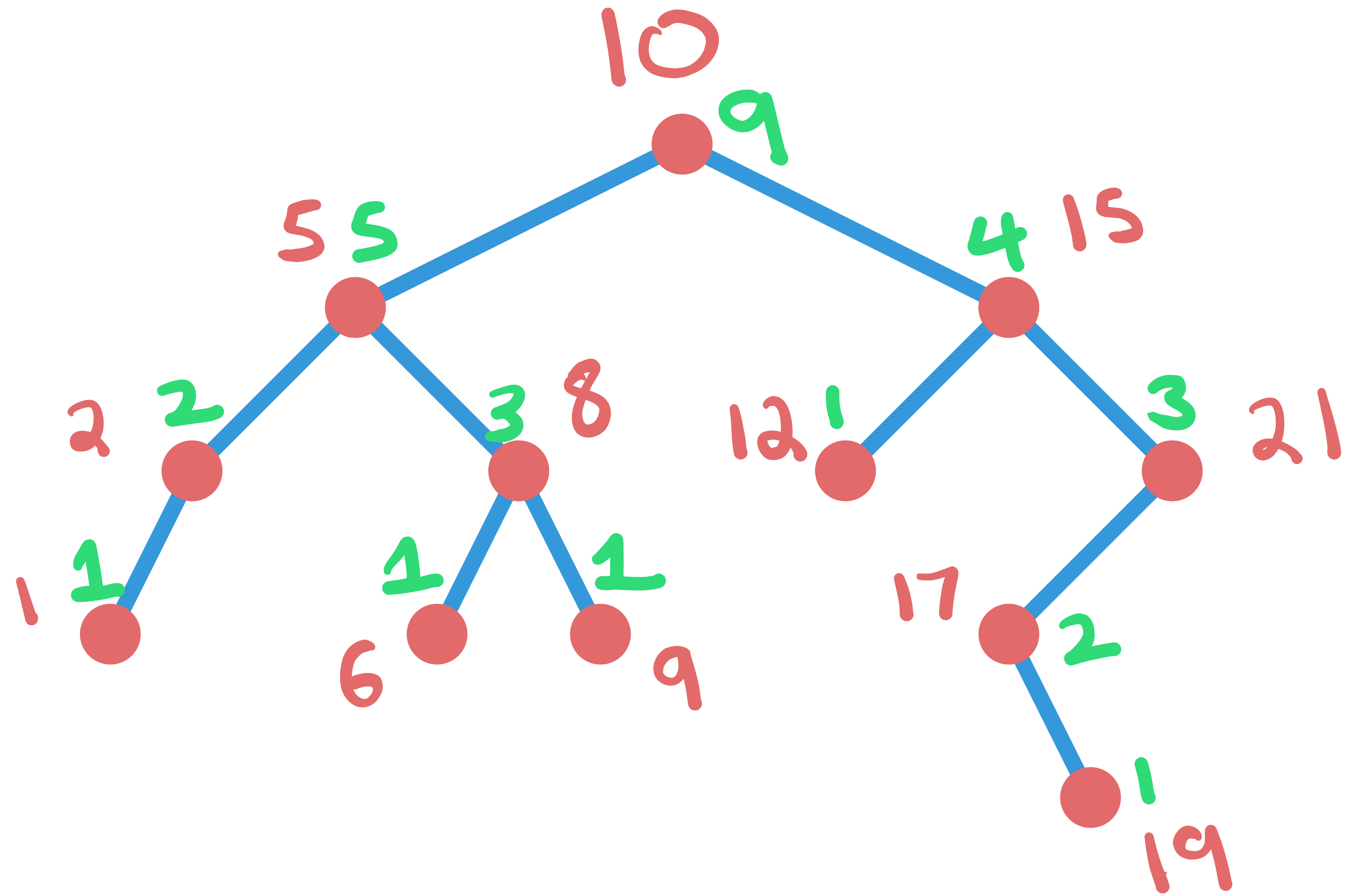
AVL trees

Red-black trees

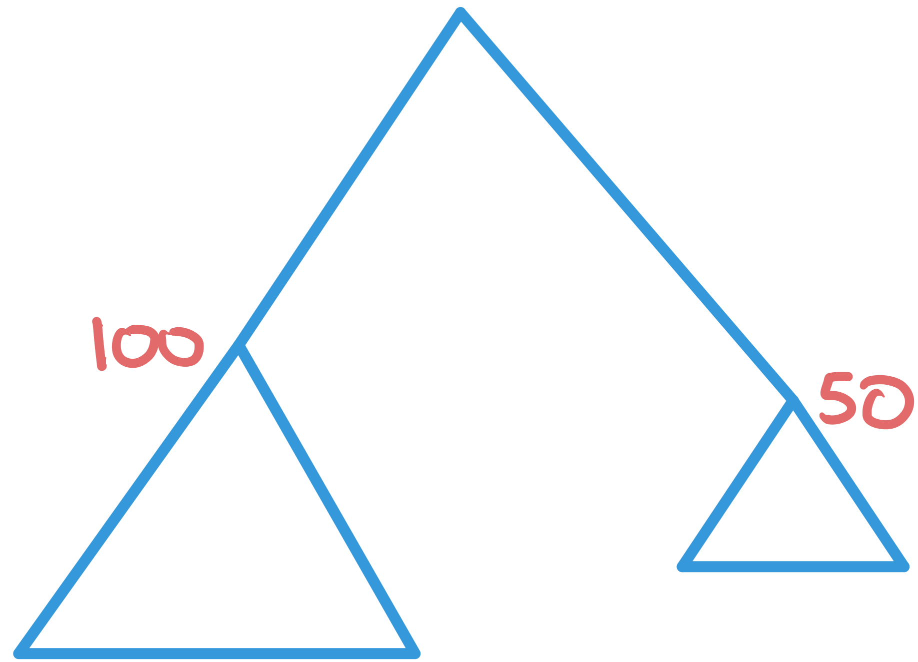
typically covered in UG data structures

Today we will be very lazy

write down # nodes in each subtree



If we detect big imbalance:



(# in one subtree) > 2×(# in other subtree),

rebuild entire subtree tree optimally

(choose median to be root of subtree, recurse) (linear time)

Lemma: height is O(log n)
(always)

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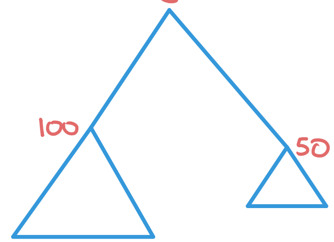

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Theorem insertion takes O(log n) amortized time

Lemma: height is $O(\log n)$ (always)

Theorem insertion takes $O(\log n)$ amortized time

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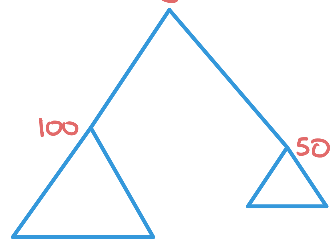

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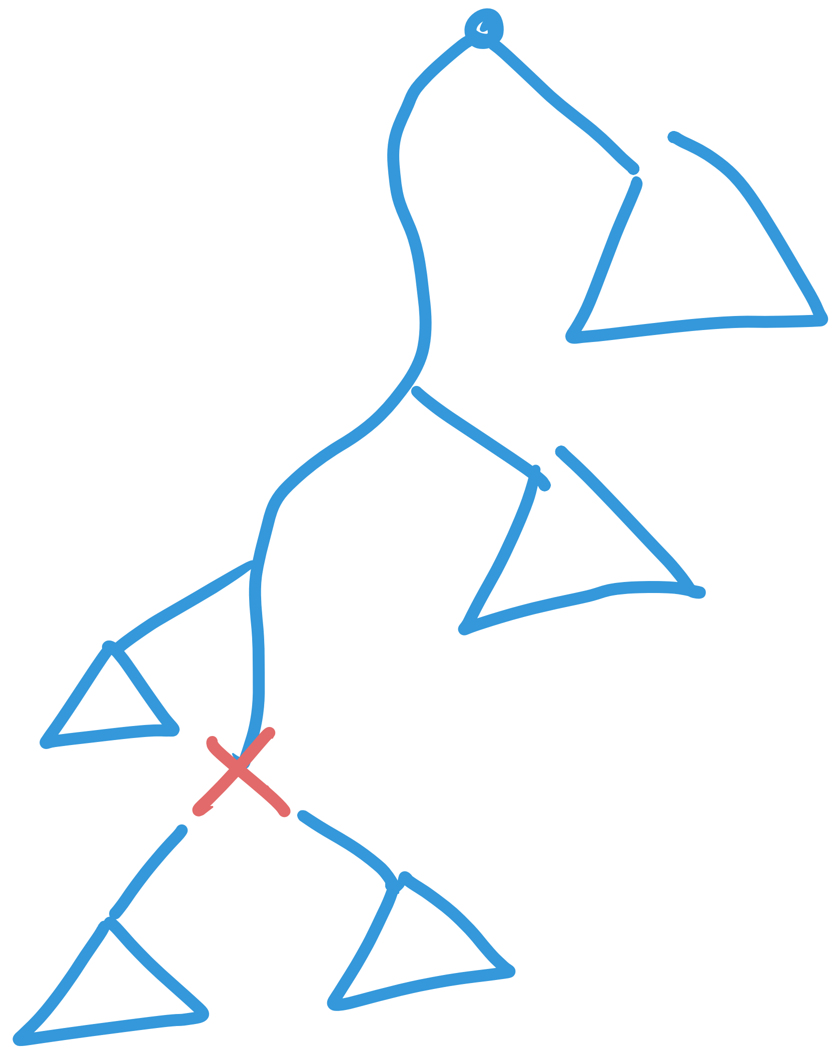
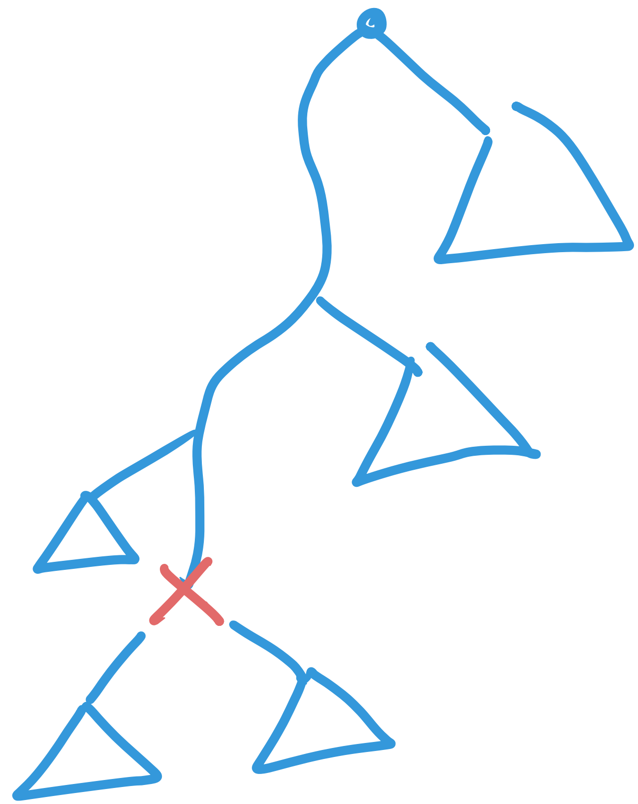
Deletions (remove key from BST)

Old-fashioned way:

① search for key

② If key is not a leaf:
chaos





Lazy idea: mark node as deleted

if $\geq$ 50% marked for deletion, rebuild optimally

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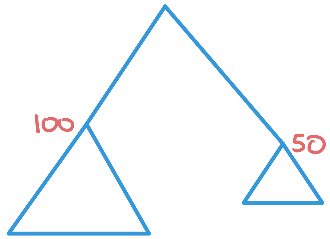

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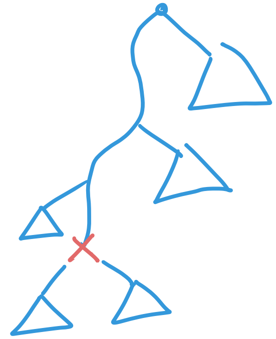

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Theorem: lazy insertion and lazy deletion take

$O(\log n)$ amortized time

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<u>Proof</u>

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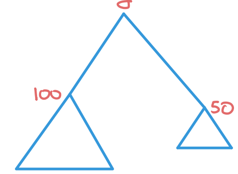
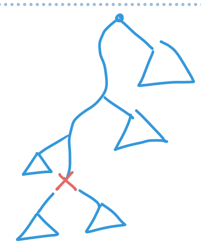


$\left(\begin{array}{c}\#\text{ in one}\\ \text{subtree}\end{array}\right) > 2\times\left(\begin{array}{c}\#\text{ in other}\\ \text{subtree}\end{array}\right)$,

rebuild entire subtree tree optimally

(choose median to be root of subtree, recurse) (linear time)



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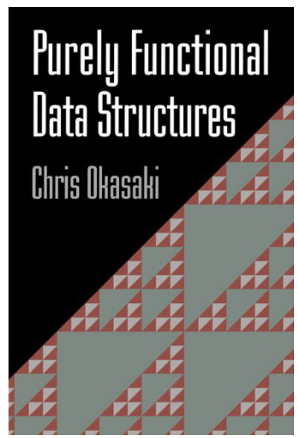
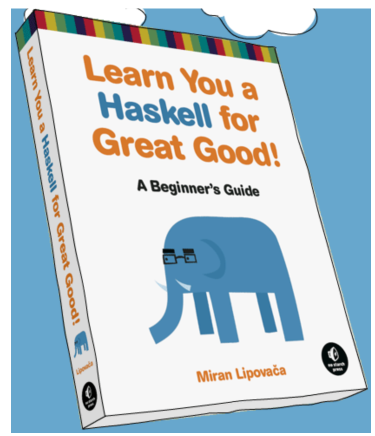
Immutable data structures

Once something is written, never changes

Restrictions bring benefits:

- no side effects
- simplifies concurrency
- helps compiler





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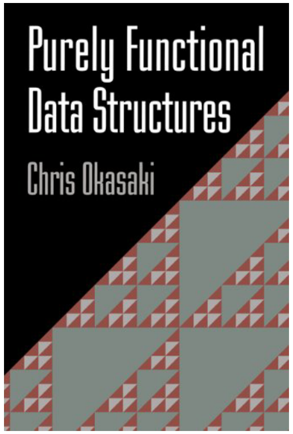
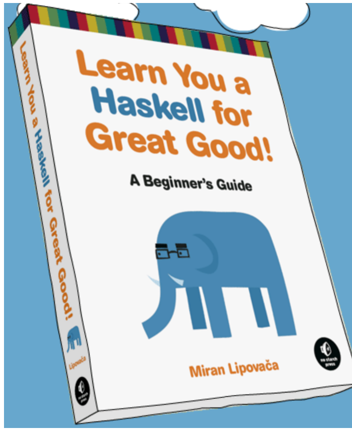
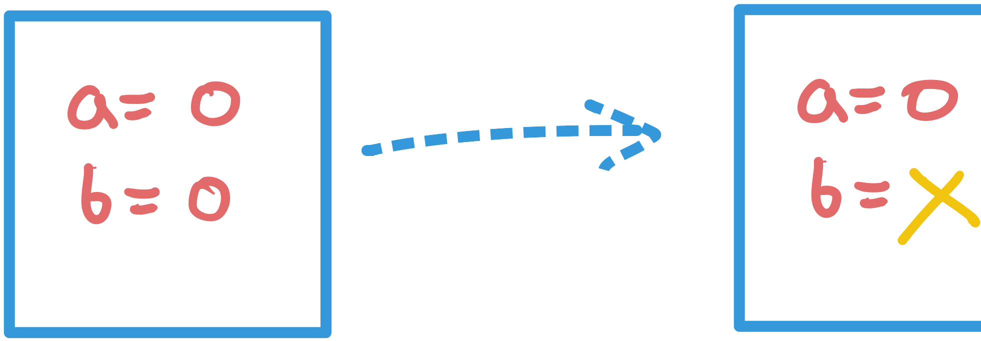




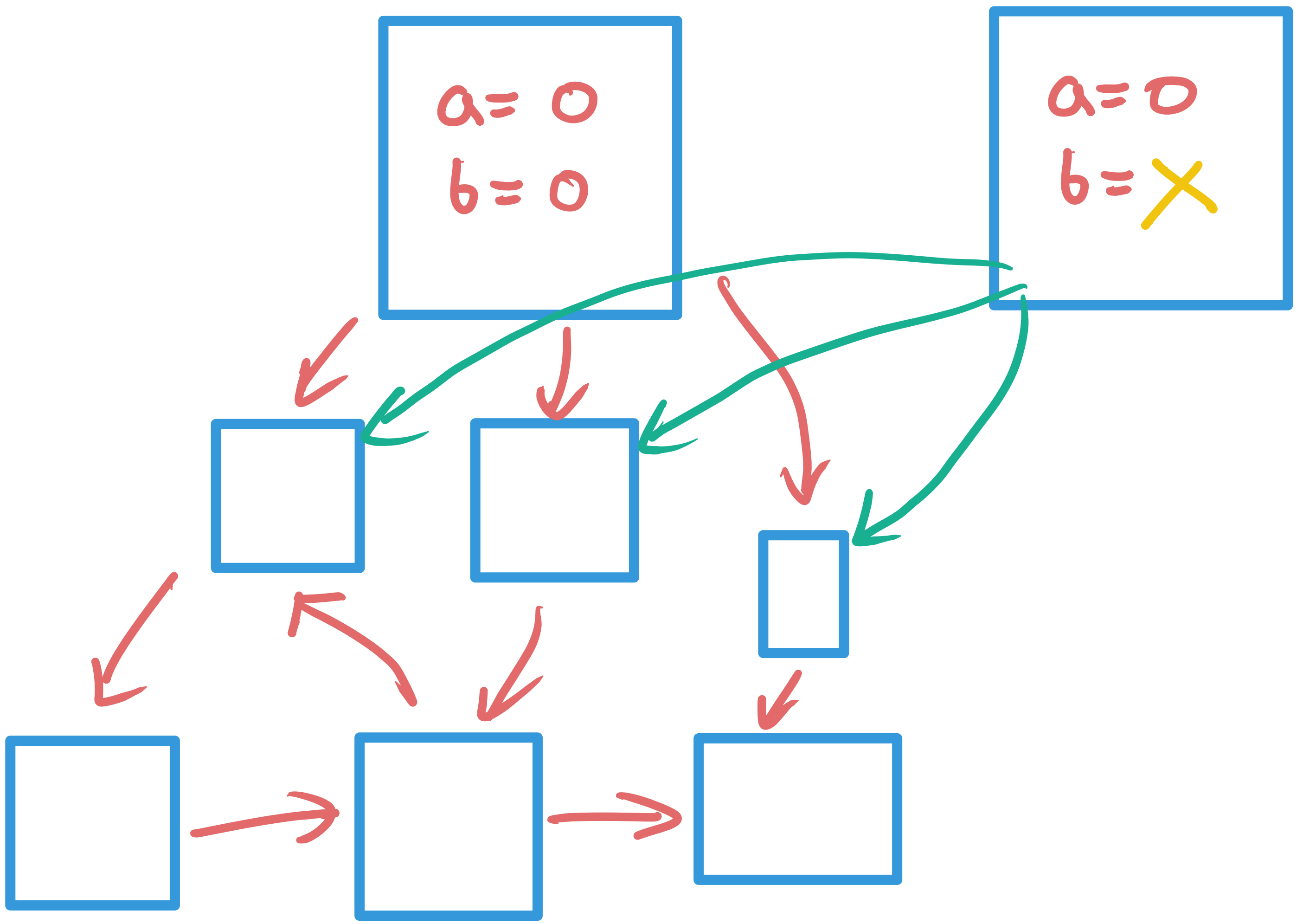
e.g. linked list

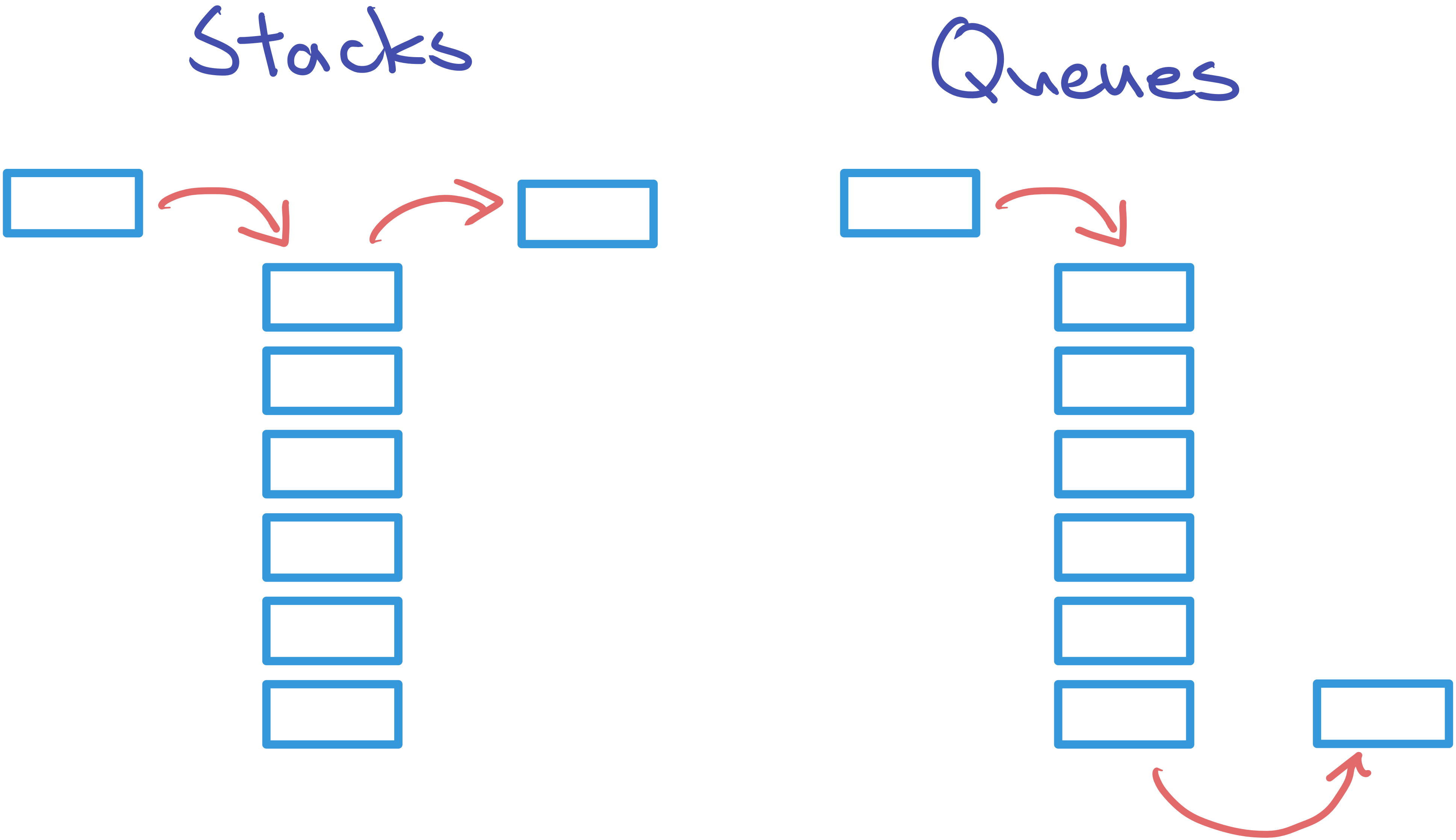
Editing objects by copying

small change ⇒ copy whole object

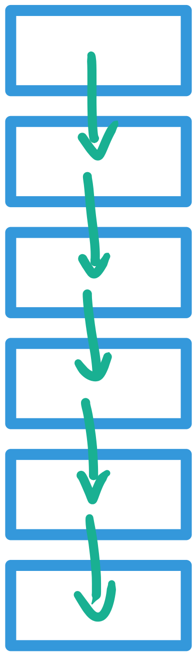


What about "deep" objects?



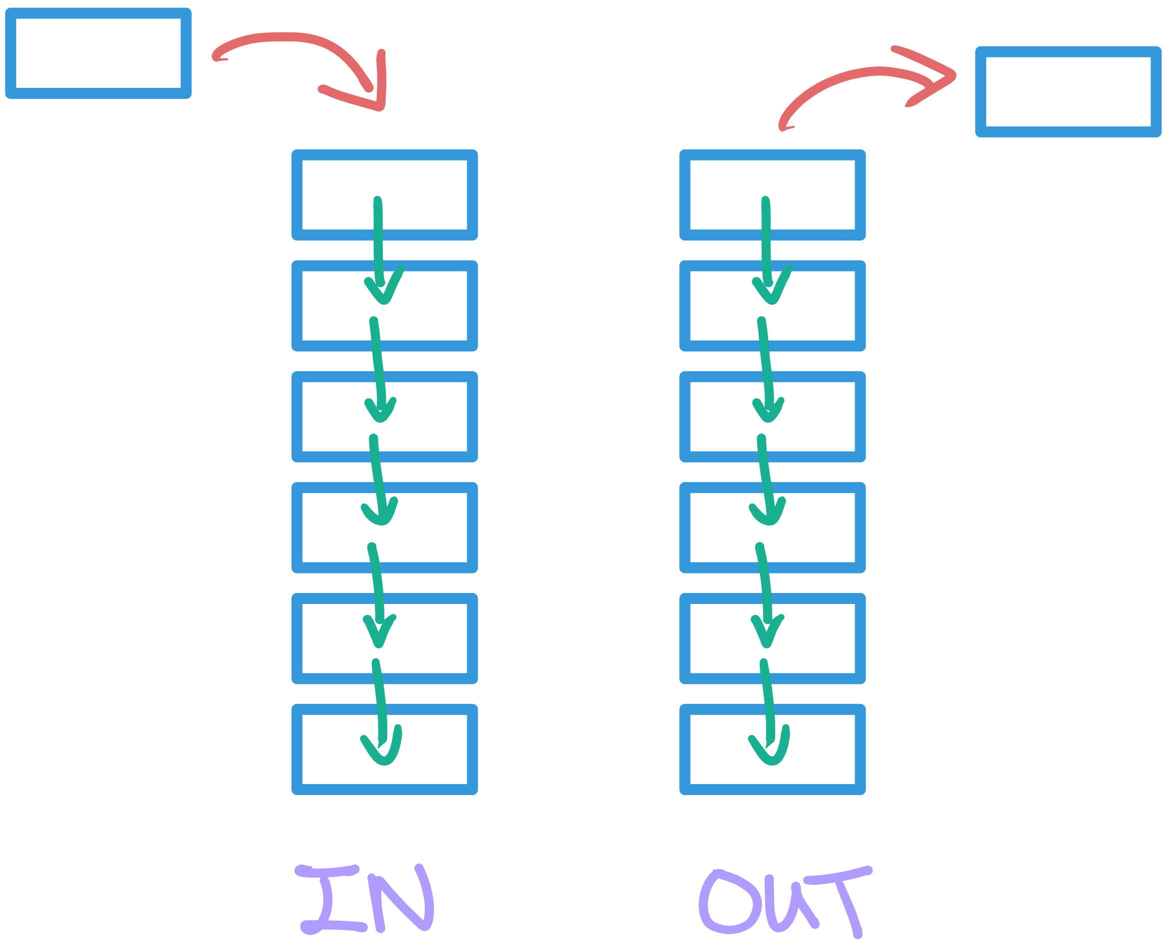

Stacks
Queues

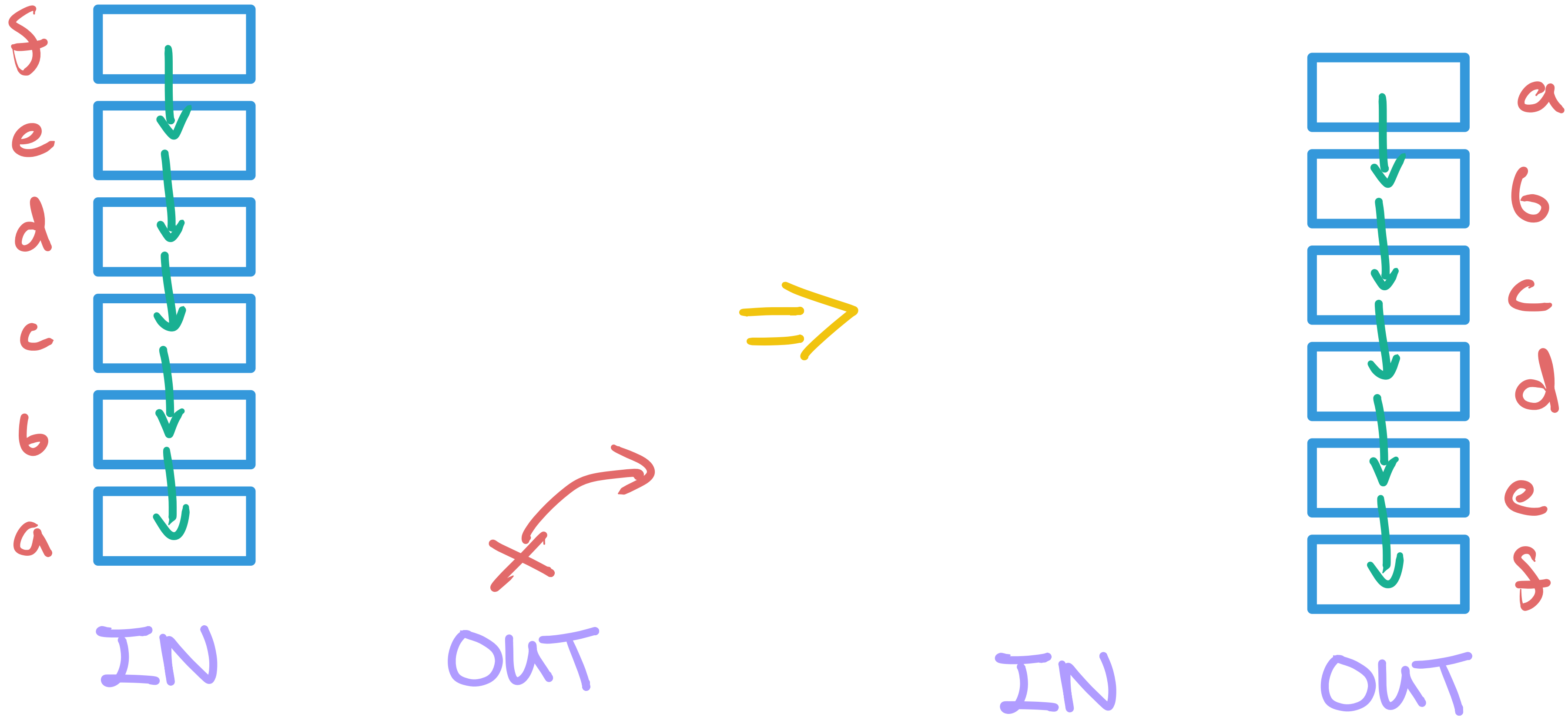
Immutable stacks



(just a linked list)

Immutable Queues





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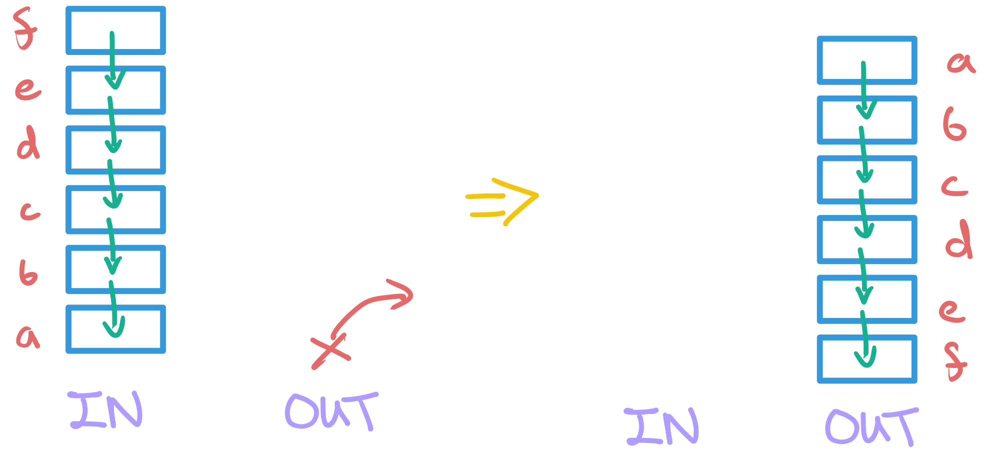

Theorem: insertion/deletion take $O(1)$ amortized